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RAMSEY PROPERTIES OF EDGE-LABELLED GRAPHS VIA COMPLETIONS

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Abstract. Motivated by applications in structural Ramsey theory, we describe "metric-like" classes of edge-labelled graphs, study their completion problems and find Ramsey expansions. They turn out to be general enough to incorporate most of the known Ramsey results for edge-labelled graphs under a common framework and also solve a problem of Conant on generalised metric spaces. As a corollary of understanding completions, one obtains homomorphism dualities for these classes.

In this note we review the correspondence between completions, structural Ramsey theory, homomorphism dualities and the constraint satisfaction problem. We apply this correspondence on certain classes of edge-labelled graphs, study their completions and as a result get a rich spectrum of Ramsey examples containing many of the previous results under one framework. We apply this to solve a problem of Conant on EPPA of generalised metric spaces [**5**].

Let L be a set. An L-edge-labelled graph is a tuple $\mathbf{G} = (G, E, d)$, where (G, E) is a (simple undirected) graph and d is a function $E \to L$. We will write $d(x, y)$ for $d(\{x, y\})$ and implicitly assume symmetry. Since E can be inferred from the domain of d, we will often treat edge-labelled graphs only as pairs (G, d). Unless explicitly stated otherwise, we only consider finite graphs

Let $\mathbf{A} = (A, d)$ and $\mathbf{B} = (B, d')$ be L-edge-labelled graphs and let $f : A \to B$ be a function. We say that f is a (label-preserving) *homomorphism* if $d(x, y) = \ell$ implies that $d'(f(x), f(y)) = \ell$ (so, in particular, $\{f(x), f(y)\}$ is an edge of $\mathbf{B}$). We will write $f : \mathbf{A} \to \mathbf{B}$ to emphasize that f respects the structure. Note that if L is a singleton set, we get the standard notion of homomorphism for graphs. Also note that if $\mathbf{A}$ and $\mathbf{B}$ are complete edge-labelled graphs, then homomorphisms coincide with the model-theoretic notion of embedding.

If $\mathbf{A}$ is an L-edge-labelled graph and $\mathbf{B}$ is a complete L-edge-labelled graph such that there is a homomorphism $f : \mathbf{A} \to \mathbf{B}$, we say that $\mathbf{B}$ is a *completion* of $\mathbf{A}$. If f is injective, we call $\mathbf{B}$ a *strong* completion of $\mathbf{A}$.

The concept of completions (which was defined in [**11**] for general structures) generalises many combinatorial problems and is interesting on its own. For example, asking whether the complete graph on 3 vertices is a completion of a graph

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is the same as asking whether a graph is 3-colourable. More generally, asking for the existence of a completion is an instance of the constraint satisfaction problem (CSP).

Example 1 (Motivating example)**.** Let L be the set of all positive integers and let $\mathcal{C}$ be the class of all complete L-edge-labelled graphs which contain no non-metric triangles, that is, triangles with labels a, b, c such that $a > b + c$. In other words, $\mathcal{C}$ can be viewed as the class of all finite metric spaces with integer distances ($d(x, x) = 0$ is understood implicitly).

A connected L-edge-labelled graph $\mathbf{A} = (A, d)$ has a completion $\mathbf{B} \in \mathcal{C}$ if and only if $\mathbf{A}$ contains no non-metric cycle (that is, a (non-induced) cycle with distances $a_0, a_1, \ldots, a_n$ such that $a_0 > \sum_{i=1}^{n} a_i$), because one can put $d' \colon \binom{A}{2} \to L$ to be the *path metric*, that is, $d'(x, y)$ is the minimum length (sum of labels) among all paths between x and y in $\mathbf{A}$, and put $\mathbf{B} = (A, d')$. Note that $\mathbf{B}$ is in fact a strong completion and furthermore $\mathrm{Aut}(\mathbf{B}) = \mathrm{Aut}(\mathbf{A})$.

Note that in the example, we in fact proved the following: Put $\mathcal{O}$ to be the class of all non-metric cycles. Then for every L-edge-labelled graph $\mathbf{A}$ we have

$$\mathcal{O} \nrightarrow \mathbf{A} \iff \mathbf{A} \to \mathcal{C},$$

where by $\mathcal{O} \nrightarrow \mathbf{A}$ we mean that there is no homomorphism from any $\mathbf{O} \in \mathcal{O}$ to $\mathbf{A}$ and by $\mathbf{A} \to \mathcal{C}$ we mean that there is $\mathbf{B} \in \mathcal{C}$ and a homomorphism $\mathbf{A} \to \mathbf{B}$. This means that $(\mathcal{O}, \mathcal{C})$ is a *homomorphism duality.* They have been studied before, in particular Nešetřil and Tadrif [**14**] classified homomorphism dualities for $\mathcal{C}$ containing a single finite structure; for more see [**7**].

We will study edge-labelled graphs from the point of view of structural Ramsey theory: A class $\mathcal{C}$ of complete L-edge-labelled graphs is *Ramsey* if for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ such that for every 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there is a copy $\mathbf{B}_0 \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}_0}{\mathbf{A}}$ is monochromatic, where by $\binom{\mathbf{U}}{\mathbf{V}}$ we mean the set of all induced subgraphs of $\mathbf{U}$ isomorphic to $\mathbf{V}$.

Hubička and Nešetřil [**11**] gave the presently strongest abstract condition for a class to be Ramsey which roughly says that understanding completions (and obstacles to completions) is very close to understanding Ramsey properties. In particular, if $\mathcal{C}$ is a *strong amalgamation class* of complete edge-labelled graphs such that there is a homomorphism-duality $(\mathcal{O}, \mathcal{C})$ with $\mathcal{O}$ finite, then the class of all linearly ordered graphs from $\mathcal{C}$ is Ramsey.

We study the completion problem via a variant of the path metric. The classes of our interest are classes of complete edge-labelled graphs determined by forbidding some edge-labelled triangles. We give explicit conditions on the sets of forbidden triangles for these classes to admit a variant of the path metric completion and apply these results to classify the obstacles (such as non-metric cycles), get a Ramsey expansion and some other combinatorial properties. Quite surprisingly, our classes can be seen as generalised metric spaces where the distances come from a partially ordered commutative semigroup. The following definitions generalise several existing concepts [**5, 3**], see [**12**].

Definition 1. A *partially ordered commutative semigroup* is a tuple $\mathfrak{M} = (M, \oplus, \preceq)$ where $\oplus$ is a commutative and associative binary operation on M, $(M, \preceq)$ is a partial order and for every $a, b, c \in \mathfrak{M}$ it holds that $a \preceq a \oplus b$, and if $b \preceq c$ then $a \oplus b \preceq a \oplus c$.

We say that an $\mathfrak{M}$-edge-labelled triangle is *non-metric* if the labels are a, b and c and it holds that $a \not\preceq b \oplus c$. More generally, we say that an $\mathfrak{M}$-edge-labelled cycle is a *non-$\mathfrak{M}$-metric cycle* if it has distances $a_0, a_1, \ldots, a_k$ such that $a_0 \not\preceq a_1 \oplus \cdots \oplus a_k$. We say that an $\mathfrak{M}$-edge-labelled graph is an $\mathfrak{M}$-metric space if it contains no non-metric triangles.

For example, the $(\mathbb{R}^{>0}, +, \leq)$-metric spaces are the standard metric spaces. A very different example are the "divisibility spaces", that is, $(\mathbb{N}^{\geq 1}, \cdot, |)$, where by $|$ we mean the "is a divisor of" relation.

For an $\mathfrak{M}$-edge-labelled path $\mathbf{P}$, we denote by $\|\mathbf{P}\|$ the $\oplus$-sum of the labels of edges of $\mathbf{P}$ and call it the $\mathfrak{M}$-*length* of $\mathbf{P}$. Using this, one can define the $\mathfrak{M}$-path metric:

Definition 2. Let $\mathfrak{M}$ be a partially ordered commutative semigroup and let $\mathbf{A} = (A, d)$ be an $\mathfrak{M}$-edge-labelled graph. We define $d' \colon \binom{A}{2} \to \mathfrak{M}$ by

$$d'(x, y) = \inf\nolimits_{\preceq}\{\|\mathbf{P}\| : \mathbf{P} \text{ is a path between } x \text{ and } y \text{ in } \mathbf{A}\}$$

and call it the $\mathfrak{M}$-path metric of $\mathbf{A}$. (If some of the infima are undefined, we let d' be undefined.)

Motivated by applications, we know that requiring $\preceq$ to be a lattice would be too strong a condition. Instead, we consider classes of $\mathfrak{M}$-metric spaces which omit homomorphic images of $\mathfrak{M}$-edge-labelled cycles from some *well-behaved* family $\mathcal{F}$ satisfying several conditions which will be given precisely in the full version of this paper [**12, 9**]. These conditions in particular ensure that the $\mathfrak{M}$-path metric is always defined for graphs omitting homomorphisms from $\mathcal{F}$ and that $\mathbf{A}$ omits homomorphisms from $\mathcal{F}$ and non-$\mathfrak{M}$-metric cycles if and only if its $\mathfrak{M}$-path metric does. Moreover, the conditions are explicit and can be directly checked given such a family $\mathcal{F}$.

In the following paragraphs, we will denote by $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ the class of all finite $\mathfrak{M}$-metric spaces containing no triangles from $\mathcal{F}$, where $\mathfrak{M}$ is a partially ordered commutative semigroup and $\mathcal{F}$ is a well-behaved family. We remark that the conditions of $\mathcal{F}$ imply that in fact $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ omits homomorphic images from $\mathcal{F}$.

We prove the following theorem:

Theorem 1. *A connected $\mathfrak{M}$-edge-labelled graph $\mathbf{A}$ has a completion in $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ if and only if it contains no (homomorphic image of a) non-$\mathfrak{M}$-metric cycle and no homomorphic image of a member of $\mathcal{F}$.*

Moreover, the $\mathfrak{M}$-path metric of $\mathbf{A}$ is a strong completion of $\mathbf{A}$ which preserves all automorphisms of $\mathbf{A}$.

A partially ordered commutative semigroup $\mathfrak{M}$ is *archimedean* if for every $a, b \in \mathfrak{M}$ there is an integer n such that $n \times a \succeq b$ (where by $n \times a$ we mean the $\oplus$-sum of n copies of a). For archimedean semigroups, we can then prove

Theorem 2. *The class of all linearly ordered graphs from* $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ *is Ramsey provided that* $\mathfrak{M}$ *is archimedean and for every finite* $S \subseteq \mathfrak{M}$ *there are only finitely many* S*-edge-labelled cycles in* $\mathcal{F}$*.*

For non-archimedean semigroups the situation is more complicated and the Ramsey expansion consists of several partial orders as in [**3**].

A class $\mathcal{C}$ of structures has the *extension property for partial automorphisms* (*EPPA*) if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$ containing $\mathbf{A}$ as a substructure (i.e. induced subgraph with labels preserved) such that every isomorphism of two finite substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$. EPPA was introduced by Hrushovski [**8**] who proved it for the class of all graphs and has been studied since, see e.g. [**16**]. Using a recent general result of the authors on EPPA [**10**], we can also prove it for $\mathfrak{M}$-metric spaces, thereby answering a question of Conant [**5**].

Theorem 3. $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ *has EPPA provided that for every finite* $S \subseteq \mathfrak{M}$ *there are only finitely many* S*-edge-labelled cycles in* $\mathcal{F}$*.*

We also get the following corollary of Theorem 1.

Corollary 4. *There is a finite family* $\mathcal{O}$ *of* $\mathfrak{M}$*-edge-labelled cycles such that* $(\mathcal{O}, \mathcal{M}_{\mathfrak{M}}^{\mathcal{F}})$ *is a homomorphism duality provided that* $\mathfrak{M}$ *is finite and archimedean and* $\mathcal{F}$ *is finite.*

It was observed by Nešetřil [**13**] that under mild (and natural) assumptions, every Ramsey class is a so-called *amalgamation class*. By the Fraïssé theorem [**6**], amalgamation classes (with at most countable L) correspond to countable homogeneous L-edge-labelled graphs – their *Fraïssé limits* – (here an edge-labelled graph $\mathbf{A}$ is *homogeneous* if every isomorphism of finite substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{A}$) and vice versa. This means that if we let $\mathbb{F}$ be the Fraïssé limit of $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ from Corollary 4, the constraint satisfaction problem for $\mathbb{F}$ is solvable in polynomial time, see [**2**].

Rather surprisingly, the classes $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ are very rich and contain many previously studied examples such as S-metric spaces [**15, 11**], Λ-ultrametric spaces [**3**] or primitive metrically homogeneous graphs [**4, 1**]. We thus get a general framework containing all the aforementioned results (and solving some open questions such as EPPA for S-metric spaces) and giving new insights into the combinatorics of many homogeneous edge-labelled graphs (for example, in an ongoing collaboration with Evans and Li we are trying to prove that the automorphism groups of the Fraïssé limits for classes with finite archimedean $\mathfrak{M}$ are simple).

This motivates the following conjecture:

Conjecture 5. Let $\mathbf{A}$ be a (countable) homogeneous L-edge-labelled graph with finite L such that the class of all finite substructures of $\mathbf{A}$ is determined by a finite set of forbidden triangles and $\mathbf{A}$ is primitive (that is, $\mathrm{Aut}(\mathbf{A})$ fixes no non-trivial partition of its vertices). Then $\mathbf{A}$ is isomorphic to the Fraïssé limit of some $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$.

A very non-trivial example where this conjecture holds are the primitive metrically homogeneous graphs, see [**12**, Chapter 6.1]. Non-examples motivating the extra conditions are bipartite graphs or affinely independent Euclidean metric spaces.

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