Acta Math. Univ. Comenianae
Vol. LXXXV, 2 (2016), pp. 173–181

QUASI-MODULAR PSEUDOCOMPLEMENTED SEMILATTICES

S. EL ASSAR, M. ATALLAH and E. ATEF

Abstract. P. Mederly [**7**] characterized the modular pseudocomplemented semilattices in terms of triples. We show that a similar result is possible for the class of quasi-modular pseudocomplemented semilattices, which is an extension of the class of modular pseudocomplemented semilattices.

1. Introduction

The triple method is one of the methods used to study structures with pseudocomplementation. The basic idea is to associate with such a structure two simpler structures (Boolean algebra and a semilattice with unit) and a connecting mapping between them, forming a triple.

In this paper, after some preliminary considerations and introducing the quasi-modular pseudocomplemented semilattices, we characterize the triples associated with these algebras. In the last section, we characterize homomorphisms and congruences of quasi-modular pseudocomplemented semilattices in terms of triples.

2. Preliminaries

A *pseudocomplemented semilattice* (PCS) is an algebra $\langle S, \wedge, ^*, 0, 1\rangle$ of type $(2,1,0,0)$, where $\langle S, \wedge, 0, 1\rangle$ is a bounded meet semilattice and for every $a \in S$, the element a^* is the *pseudocomplement* of a, i.e.,

$$x \leq a^* \qquad \text{iff} \qquad x \wedge a = 0.$$

If a PCS S forms a lattice, then it is said to be a *pseudocomplemented lattice* (PCL or p-algebra).

Let S be a PCS. The element $a \in S$ is called *closed* if $a = a^{**}$. $B(S)$ denotes the set of all closed elements of S. It is known that $\langle B(S), \veebar, \wedge, ^*, 0, 1\rangle$ forms a Boolean algebra with $a \veebar b = (a^* \wedge b^*)^*$. The element $d \in S$ is said to be *dense* if $d^* = 0$. $D(S)$ is the set of all dense elements of S. It is clear that 1 is a dense element and $D(S)$ forms a filter of S.

Received September 4, 2014; revised February 23, 2016.
2010 *Mathematics Subject Classification.* Primary 03G10, 06C05, 06D15, 03G05.
Key words and phrases. Pseudocomplemented semilattices; Quasi-modular p-algebras; Boolean algebras; Quasi-modular pseudocomplemented semilattices.

Let F be a filter of a semilattice S. By $\Theta(F)$ we mean a binary relation on S defined as: $a \equiv b\Theta(F)$ if and only if $a \wedge v = b \wedge v$ for some $v \in F$. $\Theta(F)$ is a congruence relation of S.

A semilattice $\langle S, \wedge\rangle$ is called distributive (modular) if $t \geq x \wedge y \quad (t \leq x)$ implies the existence of $x_1 \geq x$ and $y_1 \geq y$ in S such that $t = x_1 \wedge y_1$. A *distributive (modular)* PCS (p-algebra) means that the underlying semilattice (lattice) is distributive (modular). We refer to [**3**] or [**5**] for the standard results on PCS's and PCL's.

The concept of modularity, introduced by T. Katriňák and P. Mederly in [**4**], and weakened by the same authors in [**6**], for p-algebras, as follows

$$((x \wedge y) \vee z^{**}) \wedge x = (x \wedge y) \vee (z^{**} \wedge x)$$

or

$$x \geq y, \quad x \wedge (z^{**} \vee y) = (x \wedge z^{**}) \vee y.$$

The p-algebras satisfying the above identity are called *quasi-modular* (see [**6**]).

3. Construction of quasi-modular PCS's

In this section, we first introduce the decomposable PCS's and discuss their basic properties (see [**3**], [**5**], [**6**]). In the second part, we start investigating the quasi-modular pseudocomplemented semilattices.

Definition 3.1 (see [**3**], [**5**], [**6**]). A PCS S is said to be *decomposable* if for every $x \in S$, there exists $d \in D(S)$ such that

$$x = x^{**} \wedge d.$$

It is easy to verify that distributive and modular PCS's are decomposable. Let S be a decomposable PCS. For every $a \in B(S)$, a binary relation $a\overline{\varphi}(S)$ on $D(S)$ is defined by

$$d \equiv e(a\overline{\varphi}(S)) \qquad \text{iff} \qquad d \wedge a^* = e \wedge a^*.$$

It is easy to verify that $a\overline{\varphi}(S)$ is a semilattice congruence on $D(S)$ for any $a \in B(S)$. The set of semilattice congruences on $D(S)$, ordered by set inclusion, is a lattice $\mathrm{Con}(D(S))$ with smallest element $\Delta = \{(x, x) : x \in D(S)\}$ and largest one $\nabla = D(S) \times D(S)$. Clearly, $0\overline{\varphi}(S) = \Delta$ and $1\overline{\varphi}(S) = \nabla$. It is easy to verify that $a \leq b$ implies $a\overline{\varphi}(S) \subseteq b\overline{\varphi}(S)$. The mapping $a \to a\overline{\varphi}(S)$, also called the *structure mapping*, is a (0,1)-isotone mapping from $B(S)$ into $\mathrm{Con}(D(S))$. Concluding, we get $(B(S), D(S), \overline{\varphi}(S))$, the *decomposable triple* associated with S (or *d*-triple).

The triple associated with a decomposable PCS can be abstractly characterized as follows:

Definition 3.2 (see [**6**]). An *(abstract) d-triple* is $(B, D, \overline{\varphi})$, where

(i) $\langle B; \vee, \wedge, ', 0, 1\rangle$ is a Boolean algebra.

(ii) D is a $\wedge$-semilattice with 1.

(iii) $\overline{\varphi}$ is a $\{0, 1\}$ isotone mapping from B into $\mathrm{Con}(D)$.

Definition 3.3 (see [**3**]). An *isomorphism* of the d-triples $\big(B, D, \overline{\varphi}\big)$ and $\big(B_1, D_1, \overline{\varphi}_1\big)$ is a pair (f, g), where f is an isomorphism of B and B_1, and g is an isomorphism of D and D_1 such that the following diagram commutes.

$$\begin{array}{ccc} B & \longrightarrow^{\overline{\varphi}} & \mathrm{Con}(D) \\ \downarrow^{f} & & \downarrow^{\overline{g}} \\ B_1 & \longrightarrow^{\overline{\varphi}_1} & \mathrm{Con}(D_1) \end{array}$$

Here $\overline{g}$ is an isomorphism of $\mathrm{Con}(D)$ onto $\mathrm{Con}(D_1)$ assigning to each $\Theta \in \mathrm{Con}(D)$ the congruence $\Theta_1 := \overline{g}(\Theta) \in \mathrm{Con}(D_1)$ given by $\big(g(x), g(y)\big) \in \Theta_1$ if and only if $(x, y) \in \Theta$.

Definition 3.4. A PCS S is called a *quasi-modular* if $x \leq y$ *and* $x \geq y \wedge z^{**}$ in S implies the existence of $z_1 \in S$ and $z_1 \geq z^{**}$ such that $x = y \wedge z_1$.

Note that a modular PCL (PCS) is quasi-modular. We refer to [**6**] for a quasi-modular PCL-triple construction.

Lemma 3.1. *A quasi-modular PCS S is decomposable.*

Proof. Assume that $x \in S$. We shall prove that $x = x^{**} \wedge d$, where $d \in D(S)$. Since S is quasi- modular, we can consider $x \leq y$ and $x \geq y \wedge x^*$ for $x, y \in S$. By the hypothesis, there exists $d \geq x^*$ such that $x = y \wedge d$. Since $x \leq x^{**}$, we can set $y = x^{**}$, which implies $x = x^{**} \wedge d$. Therefore, $x \leq d$ and consequently, $d^* \leq x^*$. As $x^* \leq d$, we get $d^* \leq x^{**}$. Hence, $d^* \leq x^* \wedge x^{**} = 0$, which implies $d \in D(S)$. Thus, S is decomposable. □

The notion of weakly standard elements of a lattice $(L, \wedge, \vee)$ is needed, so we recall the following definition.

Definition 3.5. (see [**1**]) An element $a \in L$ is *weakly standard* if for all $x, y \in L$, $x \leq y$ implies $x \vee (a \wedge y) = (x \vee a) \wedge y$.

As a reformulation of Definition 3.5 we obtain the next lemma.

Lemma 3.2. *Let L be a PCL. Then L is quasi-modular if and only if any $a \in B(L)$ is a weakly standard element of L.*

The proof is straightforward.

Theorem 3.1. (see [**1**, Theorem 5.2.8, p. 87]) *Let $(L, \vee, \wedge)$ be a lattice. An element $a \in L$ is weakly standard if and only if there exist no $x_1, y_1 \in L$ such that $a \wedge x_1 = a \wedge y_1, x_1, a, y_1$ and $a \vee x_1 = a \vee y_1$ form a sublattice isomorphic to the pentagon N_5* (see Figure 1).

Now, we formulate the following theorem.

Theorem 3.2. *Let $L = \langle L, \wedge, \vee, ^*, 0, 1\rangle$ be a PCL. Then L is a quasi-modular PCL if and only if the reduct $L_1 = (L; \wedge, ^*, 0, 1)$, is a quasi-modular PCS.*

Proof. Let L be a quasi-modular PCL. Assume $x \leq y$ and $x \geq y \wedge z^{**}$. Put $z_1 = x \vee z^{**}$. Hence $y \wedge z_1 = y \wedge (x \vee z^{**}) = (y \wedge z^{**}) \vee x$ (by quasi-modularity of L) so we get $x = y \wedge z_1, \quad z_1 \geq z^{**}$.

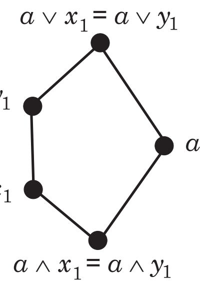

Figure 1. N_5.

Conversely, assume that L_1 is a quasi-modular PCS. Now, suppose to the contrary that L is not quasi-modular. Then there exists $z \in L$ such that $z^{**} \in B(L)$ is not weakly standard in L (see Lemma 3.2). Denote $a = z^{**}$. Hence there exist $x, y \in L$ such that $x \leq y$ and $x \vee (a \wedge y) < (x \vee a) \wedge y$ by distributive inequality, $x \leq y$ and the assumption that L is not quasi-modular. Put $x_1 = x \vee (a \wedge y)$ and $y_1 = (x \vee a) \wedge y$. Therefore, $x \leq x_1 < y_1 \leq y$ in L and the elements $a \wedge x_1 = a \wedge y_1$, x_1, a, y_1 and $a \vee x_1 = a \vee y_1$ form a sublattice that is isomorphic to N_5 (see Theorem 3.1). Clearly, by the hypothesis, $y_1 > x_1 > y_1 \wedge a$ in L_1. Thus, there exists $z_1 \in L_1$ such that $x_1 = z_1 \wedge y_1$ and $a \leq z_1$. Since L_1 is a reduct of L, the same is true in L. It is easy to verify that $a \vee x_1 \leq z_1$ in L. Since $a \vee x_1$ is the largest element of our N_5 sublattice of L, we get $x_1 = z_1 \wedge y_1 = y_1$, which is a contradiction. □

Remark 3.1. Theorem 3.2 gives an approval of Definition 3.4. It is a guarantee that Definition 3.4 is actually a good generalization of the notion "quasi-modular" to a wider class of PCS's.

Our chief aim in this section is to find a description of triples associated with quasi-modular PCS's. From [**6**, Theorem 1] and Lemma 3.1, we derive easily the following lemmas.

Lemma 3.3. *Let S and S_1 be decomposable PCS's. Assume that S is quasi-modular. Then the algebras S and S_1 are isomorphic if and only if their associated d-triples are isomorphic.*

Lemma 3.4. *Let S be a quasi-modular PCS, then*

(i) $([d) \vee [a)) \wedge D(S) = [d) \vee ([a) \wedge D(S))$ *for all* $a \in B(S)$ *and* $d \in D(S)$*;*

(ii) *S is filter-decomposable.*

Proof. (i) We recall that $[d)$, $[a)$ and $D(S)$ are filters of S, which is a PCS. It is well-known that the set of all filters of S is partially ordered by inclusion and forms a bounded lattice $F(D(S))$, the lattice of all filters of S.
Note: If $F_1,\ F_2 \in F(D(S))$, then $F_1 \wedge F_2 = F_1 \cap F_2$ (set intersection). Therefore,

$$A = ([d) \vee [a)) \wedge D(S) \supseteq [d) \vee ([a) \wedge D(S)) = B$$

by the distributive inequality. It remains to prove the reverse inclusion $A \subseteq B$. Suppose that $t \in A$. Therefore, $t = d_1 \wedge a_1$ for $d \leq d_1$ and $a \leq a_1$. Since $t \in D(S)$, we see that $a_1^{**} = 1$. On the other hand, $d_1 \in [d)$ and $a_1 \in [a) \wedge D(S)$, which implies $t \in B$, as desired. (ii) follows from (i) and [**6**, 5.1]. $\square$

Remark 3.2. Lemma 3.4 says, in other words, that for a quasi-modular PCS S, it is enough to look after an F-triple associated with S (see [**6**]). More precisely, instead of $(B(S), D(S), \overline{\varphi}(S))$ we need the F-triple $(B(S), D(S), \varphi(S))$, where $\varphi\colon B(S) \to F(D(S))$ is a structure mapping defined as follows:

$$a \to a\varphi(S) = [a^*) \cap D(S)$$

for any $a \in B(S)$ (see [**6**, Section 5]).

Lemma 3.5. *Let S be a quasi-modular PCS, and $a, b \in B$. Then*
(i) $([a) \wedge D(S)) \vee ([b) \wedge D(S)) = (([a) \wedge D(S)) \vee [b)) \wedge D(S)$,
(ii) $[b \vee a) \vee (D(S) \wedge [a)) = ([b \vee a) \vee D(S)) \wedge [a)$,
(iii) $(D(S) \vee [b)) \wedge [a \wedge b) = ((D(S) \vee [b)) \wedge [a)) \vee [b)$,
(iv) $([a) \wedge [b)) \vee D(S) = ([a) \vee D(S)) \wedge ([b) \vee D(S))$.

Proof. (i) Let $t \in (([a) \wedge D(S)) \vee [b)) \wedge D(S)$ which implies $t \in (([a) \wedge D(S)) \vee [b))$ and $t \in D(S)$. Therefore, $t \geq s \wedge b$, where $s \in [a) \wedge D(S)$. We have $s \geq t \wedge s \geq s \wedge b^{**}$. Hence $t \wedge s = s \wedge x$, $x \geq b^{**} = b$ (i.e., $x \in [b)$) by quasi-modularity. Here $x \geq t \wedge s$, and $x^* \leq (t \wedge s)^* = 0$, then $x \in [b) \cap D(S)$. Hence $t \wedge s \in ([a) \wedge D(S)) \vee ([b) \wedge D(S))$, and so

$$t \in ([a) \wedge D(S)) \vee ([b) \wedge D(S)).$$

Thus $(([a) \wedge D(S)) \vee [b)) \wedge D(S) \subseteq ([a) \wedge D(S)) \vee ([b) \wedge D(S))$. The converse inclusion is obvious.

(ii) Let $t \in ([b \vee a) \vee D(S)) \wedge [a)$ which implies $t \geq (a \vee b) \wedge s$ for some $s \in D(S)$ and $t \geq a$. We can assume $s \geq t \geq s \wedge (a \vee b)^{**}$. Hence $t = s \wedge x$, $x \geq a \vee b$ ($x \in [a \vee b)$) by quasi-modularity. Now $s \in [a)$ (since $s \geq t$) and $s \in D(S)$, then $s \in D(S) \wedge [a)$. Hence $t \in ([b \vee a)) \vee (D(S) \wedge [a))$ and $([b \vee a) \vee D(S)) \wedge [a) \subseteq ([b \vee a)) \vee (D(S) \wedge [a))$. The converse inclusion can be directly obtained.

(iii) Let $t \in (D(S) \vee [b)) \wedge [a \wedge b)$ which implies $t \geq a \wedge b$ and $t \in [b) \vee D(S)$. We have $b \geq t \wedge b \geq b \wedge a^{**}$. Hence $t \wedge b = b \wedge x$, $x \geq a$ ($x \in [a)$) by quasi-modularity. Thus $x \geq t \wedge b$ which implies $x \in ((D(S) \vee [b)) \wedge [a))$. Then $t \wedge b \in ((D(S) \vee [b)) \wedge [a)) \vee [b)$, and so

$$t \in ((D(S) \vee [b)) \wedge [a)) \vee [b).$$

Hence $(D(S) \vee [b)) \wedge [a \wedge b) \subseteq ((D(S) \vee [b)) \wedge [a)) \vee [b)$. It is easy to get the converse inclusion.

(iv) Let $t \in ([a) \vee D(S)) \wedge ([b) \vee D(S))$ which implies $t \in ([a) \vee D(S))$ and $t \in ([b) \vee D(S))$, that is, $t \geq a \wedge d_1$ and $t \geq b \wedge d_2$ for some $d_1, d_2 \in D(S)$. So $t^{**} \geq (a \wedge d_1)^{**} = a$, similarly, $t^{**} \geq b$. S is decomposable implies that $t = t^{**} \wedge d$ for some $d \in D(S)$. Since $t^{**} \in [a) \wedge [b)$, then $t \in ([a) \wedge [b)) \vee D(S)$. The converse inclusion follows easily. $\square$

Theorem 3.3. *Let S be a quasi-modular PCS. Then the structure map $\varphi(S)\colon B(S) \to F(D(S))$ is a $(0, 1, \vee)$-homomorphism.*

Proof. It is clear that $0\varphi(S) = [1)$ and $1\varphi(S) = D(L)$. Now we prove $(a \vee b)\varphi(S) = a\varphi(S) \vee b\varphi(S)$.

From Lemma 3.5 (ii) and (iv), we get

$$\begin{aligned}[b \vee a) \vee (D(S) \wedge [a)) &= (([a \vee b)) \vee D(S)) \wedge [a) = (([a) \wedge [b)) \vee D(S)) \wedge [a) \\ &= ([a) \vee D(S)) \wedge ([b) \vee D(S)) \wedge [a) \\ &= ([b) \vee D(S)) \wedge [a).\end{aligned}$$

Lemma 3.5 (iii) and the previous equality imply

$$\begin{aligned}([b) \vee D(S)) \wedge [a \wedge b) &= ((D(S) \vee [b)) \wedge [a)) \vee [b) \\ &= [b \vee a) \vee (D(S) \wedge [a)) \vee [b) = ([a) \wedge D(S)) \vee [b).\end{aligned}$$

Lemma 3.5(i) and the last equality imply

$$\begin{aligned}([a) \wedge D(S)) \vee ([b) \wedge D(S)) &= (([a) \wedge D(S)) \vee [b)) \wedge D(S) \\ &= ([b) \vee D(S)) \wedge [a \wedge b) \wedge D(S) \\ &= ([a \wedge b)) \wedge D(S) = ([a) \vee [b)) \wedge D(S).\end{aligned}$$

Therefore,

$$\begin{aligned}(a \vee b)\varphi(S) &= [a^* \wedge b^*) \wedge D(S) = ([a^*) \vee [b^*)) \wedge D(S) \\ &= ([a^*) \wedge D(S)) \vee ([b^*) \wedge D(S)) = a\varphi(S) \vee b\varphi(S).\end{aligned}$$

Thus $\varphi(S)\colon B(S) \to F(D(S))$ is a $(0, 1, \vee)$-homomorphism. □

Lemma 3.6. *Let S be a quasi-modular PCS and let $a, b, c \in B(S)$ and $d, e, f \in D(S)$. Let $b \geq a \geq b \wedge c$ and let $(b^*\varphi(S) \vee [e)) \subseteq (a^*\varphi(S) \vee [d)) \subseteq (b \wedge c)^*\varphi(S) \vee [e))$. Then*

$$a^*\varphi(S) \vee [d) = (b \wedge c_1)^*\varphi(S) \vee [e \wedge f_1)$$

for some $c_1 \in B(S)$, $f_1 \in D(S)$ and $c_1 \geq c$.

Proof. Put $x = a \wedge d$, $y = b \wedge e$ and $z = c \wedge f$ elements from S. Since S is filter decomposable, then by [**6**, 5.1] and the hypothesis, we obtain $[b \wedge e) \subseteq [a \wedge d) \subseteq [(b \wedge c) \wedge e)$. Hence $y \geq x \geq y \wedge z^{**}$. By quasi-modularity of S,

$$x = y \wedge z_1, \quad z_1 \geq z^{**}.$$

Put $z_1 = c_1 \wedge f_1$. Therefore, $[x) \wedge D(S) = [y \wedge z_1) \wedge D(S)$, that is, $[a \wedge d) \wedge D(S) = [(b \wedge c_1) \wedge (e \wedge f_1)) \wedge D(S)$. Again by [**6**, 5.1],

$$a^*\varphi(S) \vee [d) = (b \wedge c_1)^*\varphi(S) \vee [e \wedge f_1)$$

as required. □

Now, the triple associated with a quasi-modular PCS can be defined as follows

Definition 3.6. A triple $\langle B, D, \varphi\rangle$ is said to be an *F-triple of quasi-modular PCS* if

(i) $\langle B; \vee, \wedge, ', 0, 1\rangle$ is a Boolean algebra.

(ii) $\langle D; \wedge, 1\rangle$ is a semi-lattice with 1.

(iii) φ is a $\{0, 1, \vee\}$ homomorphism from B into $F(D)$.

(iv) If $a, b, c \in B$ and $d, e, f \in D$. Let $b \geq a \geq b \wedge c$ and let $(b'\varphi \vee [e)) \subseteq (a'\varphi \vee [d)) \subseteq ((b \wedge c)'\varphi \vee [e))$. Then

$$a'\varphi \vee [d) = (b \wedge c_1)'\varphi \vee [e \wedge f_1)$$

for some $c_1 \in B$, $f_1 \in D$ and $c_1 \geq c$.

Theorem 3.4 (Construction). *Let $\langle B, D, \varphi\rangle$ be an F-triple of quasi-modular PCS's, then we can construct a quasi-modular PCS S such that the triples $\langle B(S), D(S), \varphi(S)\rangle$ and $\langle B, D, \varphi\rangle$ are isomorphic.*

Proof. Considering the representation given in [**6**],

$$S := \{(a, a'\varphi \vee [d)) : a \in B,\ d \in D\}$$

is a decomposable PCS such that $\langle B(S), D(S), \varphi(S)\rangle \cong \langle B, D, \varphi\rangle$.

It remains to prove the quasi-modularity of S. Put $x := (a, a'\varphi \vee [d))$, $y := (b, b'\varphi \vee [e))$ and $z := (c, c'\varphi \vee [f))$. Let $y \geq x \geq y \wedge z^{**}$, then $(b, b'\varphi \vee [e)) \geq (a, a'\varphi \vee [d)) \geq (b \wedge c, (b \wedge c)'\varphi \vee [e))$ implies $b \geq a \geq b \wedge c = b \wedge c''$, and $b'\varphi \vee [e) \subseteq a'\varphi \vee [d) \subseteq (b \wedge c)'\varphi \vee [e)$. Therefore, $a'\varphi \vee [d) = (b \wedge c_1)'\varphi \vee [e \wedge f_1)$ for some $c_1 \in B$, $f_1 \in D$ and $c_1 \geq c$. Hence

$$(a, a'\varphi \vee [d)) = (b \wedge c_1, (b \wedge c_1)'\varphi \vee [e \wedge f_1)),$$

that is, $x = y \wedge z_1$, $z_1 = (c_1, c_1'\varphi \vee [f_1)) \geq (c, c'\varphi) = z^{**}$ as required. □

Concluding, from Lemma 3.3 and Theorem 3.4, we get the following theorem.

Theorem 3.5. *Two quasi-modular pseudocomplemented semilattices are isomorphic if and only if the associated triples are isomorphic. Every F-triple of quasi-modular PCS is isomorphic to a triple associated with a quasi-modular pseudocomplemented semilattice.*

4. Homomorphisms and congruence relations

The results in this section are direct consequences of [**6**, Section 7].

Let S, S_1 be quasi-modular PCS's. A mapping $h\colon S \to S_1$ is said to be a homomorphism if h preserves the operations $\wedge$ and *.

Considering Lemma 3.1 and [**6**, 7.1], we get the following theorem.

Theorem 4.1. *Let S and S_1 be quasi-modular PCS's and let $h\colon S \to S_1$ be a homomorphism. Then the restriction $h_B = h|B(S)$ is a Boolean homomorphism of $B(S)$ into $B(S_1)$ and the restriction $h_D = h|D(S)$ is a homomorphism of $D(S)$ into $D(S_1)$ that preserves 1. Moreover, h is onto if and only if h_B and h_D are onto.*

Definition 4.1. Let (B, D, φ) and (B_1, D_1, φ_1) be F-triples of quasi-modular PCS's. $(f, g)\colon (B, D, \varphi) \to (B_1, D_1, \varphi_1)$ is a PCS F-triple homomorphism if $f\colon B \to B_1$ is a Boolean homomorphism, $g\colon D \to D_1$ is a 1-preserving homomorphism such that for every $a \in B(S)$,

$$a\varphi g \subseteq af\varphi_1$$

holds.

Theorem 4.2. *Let S and S_1 be quasi-modular PCS's. If* $(f,g)\colon (B(S),D(S),\varphi(S)) \to (B(S_1),D(S_1),\varphi(S_1))$ *is an F-triple of quasi-modular PCS homomorphism, then there exists a unique homomorphism* $h\colon S \to S_1$ *such that* $h_B = f$ *and* $h_D = g$.

Proof. The proof can be given by using Lemma 3.4 and [**6**, 7.3]. □

Definition 4.2. Let (B,D,φ) be an F-triple of quasi-modular PCS. A congruence relation of (B,D,φ) is a pair (θ_1,θ_2), where θ_1 is a congruence relation of B, θ_2 is a congruence relation of D and $a \equiv b\ (\theta_1)$ implies $[a\varphi]\theta_2 = [b\varphi]\theta_2$ for any $a,b \in B$, where $[c\varphi]\theta_2 = \{x \in D:\ \text{there exists } y \in c\varphi \textit{ with } x \equiv y(\theta_2)\}$.

The characterization of congruence relations on quasi-modular PCS's is described by the following theorem.

Theorem 4.3. *Let S be a quasi-modular PCS. If θ is a congruence relation on S, then (θ_B,θ_D) is a congruence relation of $(B(S),D(S),\varphi(S))$, where θ_B and θ_D are restrictions of θ to $B(S)\times B(S)$ and $D(S)\times D(S)$, respectively. Conversely, let (θ_1,θ_2) be a congruence relation of $(B(S),D(S),\varphi(S))$. Then there exists a uniquely determined congruence relation θ of S with $\theta_B=\theta_1$ and $\theta_D=\theta_2$ such that*

$$x = x^{**}\wedge d \equiv y = y^{**}\wedge e(\theta) \qquad \textit{iff} \qquad x^{**}\equiv y^{**}(\theta_1) \quad \textit{and} \quad d \equiv e(\theta_2).$$

Proof. Since S is filter decomposable by Lemma 3.4, the proof is straightforward as in [**6**, 7.4]. □

Acknowledgment. The authors would like to thank the referees for their useful comments due to which the paper is improved to great extent.

References

1. Chajda I., Halaš R. and Kühr J., *Semilattice Structures*, Research and Exposition in Mathematics, Vol. 30, Heldermann Verlag, 2007.
2. Chen C. C., and Grätzer G., *Stone lattices I: Construction theorem*, Canad. J. Math. **21** (1969), 884–894.
3. Grätzer G., *Lattice Theory: Foundation*, Birkhäuser 2011.
4. Katriňák T., and Mederly P., *Constructions of modular p-algebras*, Algebra Universalis **4** (1974), 301–315.
5. Katriňák T., *p-algebras*, Colloq. Math. Soc. J. Bolyai, Szeged, **33** (1980), 549–573.
6. Katriňák T., and Mederly P., *Constructions of p-algebras*, Algebra Universalis **17** (1983), 288–316.
7. Mederly P., *A Characterization of modular pseudocomplemented semilattices*, Colloq Math. Soc. J. Bolyai, Szeged, **14** (1974), 231–248.

S. El Assar, Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt, *e-mail*: `selassar@yahoo.com`

M. Atallah, Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt, *e-mail*: `atallahm@hotmail.com`

E. Atef, Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt, *e-mail*: `eman60145@yahoo.com`