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ON THE DENSITY OF C_7-CRITICAL GRAPHS

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Abstract. In 1959, Grötszch famously proved that every planar graph of girth at least 4 admits a homomorphism to C_3. A natural generalization is the following conjecture: for every positive integer t, every planar graph of girth at least $4t$ admits a homomorphism to C_{2t+1}. This is the planar dual of a well-known conjecture of Jaeger, which states that every $4t$-edge-connected graph admits a modulo $(2t+1)$-orientation. Though Jaeger's original conjecture was recently disproved, it has been shown to hold for $6t$-edge-connected graphs. This implies that every planar graph of girth at least $6t$ admits a homomorphism to C_{2t+1}. We improve upon the $t = 3$ case, by showing that every planar graph of girth at least 16 admits a homomorphism to C_7. We obtain this through a more general result regarding the density of critical graphs: if G is a C_7-critical graph with $G \notin \{C_3, C_5\}$, then $e(G) \geq \frac{17v(G)-2}{15}$. Our girth bound is the best known result for Jaeger's Conjecture in the $t = 3$ case.

1. Introduction

In 1951, Dirac [**2**] introduced the concept of colour-criticality and since then, colour-critical graphs have been widely studied. A graph G is *k-critical* if its chromatic number is k and the chromatic number of every proper subgraph of G is strictly less than k. As every graph with chromatic number k contains a k-critical subgraph, it is useful to study k-colourability via colour-critical graphs. More generally, it is useful to study graph homomorphisms[1] through homomorphism-critical graphs, which we define as follows.

Definition 1.1. Let H be a graph. A graph G is *H-critical* if every proper subgraph of G admits a homomorphism to H, but G itself does not.

Perhaps one of the more famous results concerning homomorphisms of planar graphs is Grötszch's Theorem [**5**], which states that every planar graph of girth at least 4 admits a homomorphism to C_3 (or equivalently, is 3-colourable). As a natural generalization of this, one might conjecture the following.

Conjecture 1.2. If G is a planar graph of girth at least $4t$, then G admits a homomorphism to C_{2t+1}.

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[1]A homomorphism $\phi\colon G \to H$ from a graph G to a target graph H is a mapping of the vertices of G to those of H, such that for each edge $uv \in E(G)$, $\phi(u)\phi(v) \in E(H)$.

This is in fact the planar dual of a well-known conjecture of Jaeger [**6**] which states that every $4t$-edge-connected graph admits a modulo $(2t+1)$-orientation[2]. Though Jaeger's original conjecture was shown to be false in early 2018 [**4**], all counterexamples found thus far are non-planar. As such, Conjecture 1.2 is still open. We note that Conjecture 1.2 is equivalent to saying that if G is a planar graph of girth at least $4t$, then G admits a $\frac{2t+1}{t}$-circular colouring. For an overview on circular colouring, see [**14**]. The $t=1$ case is the only case in which the conjecture has been confirmed: it is equivalent to Grötzsch's theorem, which states that every triangle-free planar graph is 3-colourable.

Considerable progress has been made in the general t case, though the girth bound of $4t$ remains elusive. In 1996, Nešetřil and Zhu [**11**] showed that every planar graph of girth at least $10t-4$ admits a homomorphism to C_{2t+1}. In 2000, Klostermeyer and Zhang [**7**] showed that it was sufficient to bound the odd girth[3] of the graph as being at least $10t-3$. A year later, Zhu [**15**] showed that a girth of at least $8t-3$ is sufficient, and in 2003, Borodin et al. [**1**] improved upon this by showing a girth of at least $\frac{20t-3}{3}$ suffices. Progress stalled for a decade until in 2013, Lovász et al. [**10**] showed that every $6t$-edge-connected graph admits a modulo $(2t+1)$-orientation. As a corollary to this, they obtain that every planar graph of girth at least $6t$ admits a homomorphism to C_{2t+1}. This is the best known general bound, though in the $t=2$ case Dvořák and Postle [**3**] showed that every planar graph of odd girth at least 11 (and hence of girth at least 10) admits a homomorphism to C_5.

Our first main result is the following.

Theorem 1.3. *If G is a planar graph with girth at least 16, G admits a homomorphism to C_7.*

This stems from Theorem 1.4, below, in which we bound the density of C_7-critical graphs. A trivial density bound for C_{2t+1}-critical graphs arises from the fact that they have minimum degree two. This tells us that if G is a C_{2t+1}-critical graph, then $e(G) \geq v(G)$. Unfortunately, we cannot beat this bound in the general case as for $t \geq 1$, the $(2t-1)$-cycle is C_{2t+1}-critical. However, we can do better if we assume G contains a vertex of degree at least 3. In this case, a straightforward discharging argument shows that if G is a C_{2t+1}-critical graph that contains a vertex of degree at least 3, then $e(G) \geq \left(1+\frac{1}{4t}\right)v(G)+\frac{1}{3t}$.

Ours are not the first density results regarding C_{2t+1}-critical graphs. In [**1**], Borodin et al. show that if G is a C_{2t+1}-critical graph with girth at least $6t-2$, then G contains a subgraph H with $e(H) \geq (1+\frac{3}{10t-4})v(H)$. In [**3**], Dvořák and Postle give the best-known result for the $t=2$ case by showing that every C_5-critical graph on at least four vertices has $e(G) \geq \frac{5v(G)-2}{4}$.

Our second main result, which concerns the density of C_7-critical graphs, is the following.

[2]That is, an orientation of its edges such that for each vertex, the difference of the in-degree and the out-degree is congruent to 0 modulo $2t+1$.
[3]The *odd girth* of a graph is the length of its shortest odd cycle.

Theorem 1.4. *Let G be a C_7-critical graph. If $G \notin \{C_3, C_5\}$, then $e(G) \geq \frac{17v(G)-2}{15}$.*

This gives the best known density result for C_7-critical graphs. From Theorem 1.4 and using Euler's formula for graphs embedded in surfaces, we immediately obtain the following result.

Theorem 1.5. *If G is a planar or projective planar graph of girth at least 17, then G admits a homomorphism to C_7.*

In order to further lower the girth bound to 16 in the planar case and obtain Theorem 1.3, we use the following lemma of Klostermeyer and Zhang [**7**].

Lemma 1.6 (Folding Lemma)**.** *Let G be a planar graph with odd girth k. If $C = v_0 \dots v_{r-1} v_0$ is a cycle in G that bounds a face and $r \neq k$, then there is an integer $i \in \{0, \dots, r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} $(\mathrm{mod}\ r)$ is of odd girth k.*

With this, we obtain from Theorem 1.5 the following theorem:

Theorem 1.7. *If G is a planar graph with odd girth at least 17, then G admits a homomorphism to C_7.*

Proof. By the Folding Lemma, we may assume a minimum counterexample to Theorem 1.7 only has faces of length 17. The theorem now follows directly from Theorem 1.4 and Euler's formula for planar graphs. □

In Section 2, we will highlight some of the more important concepts used in the proof of Theorem 1.4. For simplicity and brevity, many structural lemmas are omitted; those that are included are included without proof. Section 3 gives an overview of how the discharging portion of the proof unfolds, and of how the structure and techniques described in Section 2 are used.

2. Preliminaries

Before proceeding with an outline of the proof of Theorem 1.4, we will introduce some of the concepts and techniques used. A crucial part of our analysis of graph homomorphisms consists of examining the extensions of partial homomorphisms to the entire graph. Paths with internal vertices of degree 2 play an important role in our investigation, as it is easy to determine the extensions of a partial homomorphism along such paths. In addition, the low-density C_7-critical graphs we study contain a relatively high amount of vertices of degree two. As a consequence, such paths are ubiquitous. For these reasons, we define the following terms.

Definition 2.1. A *string* in a graph G is a path with internal vertices of degree two and endpoints of degree at least three. A *k-string* is a string with k internal[4] vertices. We say a vertex is *incident with a string* if it is an endpoint of the string. Two vertices *share* a string if they are the endpoints of that string.

[4]A vertex v in a path P is *internal* if it is not an endpoint of P.

If a vertex is incident with many long strings, then its local density is relatively low. As we aim to lower-bound the density of C_{2t+1}-critical graphs, it is useful to be able to bound the number of degree two vertices in the strings incident with vertices of degree at least three. This is accomplished in the following lemma.

Lemma 2.2. *If G is an C_{2t+1}-critical graph, then G does not contain a k-string with $k \geq 2t-1$.*

The proof of this lemma stems from the fact that if P is a path with $2t-1$ vertices, $S = uPv$ is a $2t-1$-string in G, and ϕ is a homomorphism from $G \setminus P$ to C_{2t+1}, then no matter $\phi(u)$ and $\phi(v)$, there exists an extension of ϕ to G. This contradicts the fact that G is C_{2t+1}-critical.

Lemma 2.2 gives us a bound on the local density surrounding a vertex of degree at least three, but a better bound arises by considering the entire set of strings incident with the vertex rather than each string individually. To that end, we define the *weight* of a vertex as follows.

Definition 2.3. Let G be a graph, and let $v \in V(G)$ be a vertex of degree $d \geq 3$, and let $k_1, k_2, \ldots, k_d$ be integers with $k_1 \geq \cdots \geq k_d$. If v is incident with d distinct strings $S_1, \ldots, S_d$ where S_i is a k_i-string for each $1 \leq i \leq d$, we say v is of *type* $(k_1, \ldots, k_d)$. If v is a vertex of type $(k_1, \ldots, k_d)$, we define the *weight of* v as $\mathrm{wt}(v) = \sum_{i=1}^{d} k_i$.

Note that if G is a C_{2t+1}-critical graph, then G is 2-connected. Thus no vertex in a C_{2t+1}-critical graph is both endpoints of a single string, and hence the type of a vertex in a C_{2t+1}-critical graph is always well-defined. We bound the weight of vertices as follows.

Lemma 2.4. *Let G be an C_{2t+1}-critical graph. If $v \in V(G)$, then $wt(v) \leq (2t-1)\deg(v) - (2t+1)$.*

Similarly to the proof of Lemma 2.2, the proof of this lemma follows by noting that if a vertex $v \in V(G)$ has weight at least $(2t-1)\deg(v) - (2t+1)$, then there exists a homomorphism $\phi\colon G \setminus \{v\} \to C_{2t+1}$ that extends to G.

Cycles of length seven play an important role in establishing the structure of C_7-critical graphs in the proof of Theorem 1.4. We call a $(2t+1)$-cycle in a C_{2t+1}-critical graph a *cell*. In the discharging portion of our proof, cells aggregate and dispense charge in the graph in much the way vertices do. In this way, we think of cells as elementary structures, and treat them as supervertices. It is therefore unsurprising that notions of cell *weight*, *degree* and *type* (analogous to their vertex counterparts) prove useful in our analysis.

In the spirit of Lemma 2.4, the following lemma provides some restriction on the local structure surrounding a cell in a C_{2t+1}-critical graph.

Lemma 2.5. *Let G be a C_{2t+1}-critical graph. If C is a cell of G, then $wt(C) \leq (2t-1)\deg(C) - (2t+1)$.*

Several other lemmas are needed in order to rule out the existence of certain types of vertices in C_7-critical graphs and to establish the local structure surrounding others. These lemmas are omitted for simplicity.

In the proof of Theorem 1.4, we will use *potential* to learn about the density of subgraphs of minimum counterexample to Theorem 1.4. The potential method used here was popularized by Kostochka and Yancey in [**9**] in order to give a lower bound on the number of edges in colour-critical graphs. In [**3**], Dvořák and Postle use potential to bound the density of C_5-critical graphs. For our purposes, the *potential* of a graph G is given by $p(G) = 17v(G) - 15e(G)$. We note that potential on its own is merely a measure of the density of the graph: what makes it a powerful tool for structural analysis is a reduction technique which allows us to bound potential of subgraphs of a critical graph using the potential of the critical graph itself.

3. Outline of the proof of Theorem 1.4

3.1. Structure of a minimum counterexample

The proof of Theorem 1.4 is obtained via reducible configurations and discharging. For the remainder of our discussion, G refers to a counterexample to Theorem 1.4 with $v(G)$ minimum and, subject to that, with $e(G)$ minimum. The following lemma gives us insight into the potential of subgraphs of G.

First, we require the following definition.

Definition 3.1. Let H be a graph. We denote by $P_t(H)$ the set of graphs obtained from H by adding a path P of length t joining two distinct vertices of H, such that the internal vertices of P are disjoint from $V(H)$.

Lemma 3.2. *Let H be a subgraph of G. Then the following all hold:*

1. $p(H) \geq 3$ *if* $H = G$,
2. $p(H) \geq 10$ *if* $G \in P_5(H)$,
3. $p(H) \geq 12$ *if* $G \in P_4(H)$,
4. $p(H) \geq 14$ *if* $G \in P_3(H)$,
5. $p(H) = 14$ *if* $H = C_7$, *and*
6. $p(H) \geq 15$ *otherwise.*

This lemma is used in establishing the structure of G, which will be useful in the discharging portion of the proof. More specifically, using Lemma 3.2 we characterize the intersection of certain cycles in G: if C and C' are distinct 7-cycles in G, then C and C' are vertex-disjoint; if C and C' are cycles of length seven and nine, respectively, in G, then C and C' are edge-disjoint; if C and C' are distinct 9-cycles in G, with $V(C) \cap V(C') \neq \emptyset$, then their intersection is a path of length at most two.

3.2. Discharging

Recall G is a counterexample to Theorem 1.4 with $v(G)$ minimum and, subject to that, with $e(G)$ minimum. Since potential is integral, it follows that $p(G) \geq 3$. Let $X \subseteq V(G)$ be the set of vertices of degree at least three. We assign an initial charge of $ch_0(v) = 15\deg(v) - 2\mathrm{wt}(v) - 34$ to each vertex $v \in X$, and $ch_0(v) = 0$ for each $v \in V(G) \setminus X$. Note $\sum_{v\in X}(15\deg(v) - 2\mathrm{wt}(v) - 34) =$

$\sum_{v \in X}(15\deg(v) - 34) - \sum_{v \in V(G) \setminus X} 4$, since every vertex v of degree 2 contributes to the weight of two distinct vertices in X (namely, the endpoints of the string that contain v). Since $\sum_{v \in V(G) \setminus X} 4 = \sum_{v \in V(G) \setminus X} 34 - 15\deg(v)$, we have

$$
\begin{aligned}
\sum_{v \in V(G)} ch_0(V) &= \sum_{v \in V(G)} (15\deg(v) - 34) \\
&= 15 \sum_{v \in V(G)} \deg(v) - \sum_{v \in V(G)} 34 \\
&= 30e(G) - 34v(G) \\
&= -2p(G) \\
&\leq -6, \text{ since } p(G) \geq 3 \text{ and potential is integral.}
\end{aligned} \tag{1}
$$

We will redistribute the charge among the vertices and cells until every vertex and cell has non-negative charge, contradicting the fact that the sum of the charges is at most -6.

For simplicity, we will refer to a vertex as being *poor* if it has negative charge. Note by Lemma 2.4, if v is a vertex in $V(G)$, then $\mathrm{wt}(v) \leq 5\deg(v) - 7$. For a vertex $v \in X$, we therefore have $ch_0(v) \geq 15\deg(v) - 2(5\deg(v) - 7) - 34 = 5\deg(v) - 20$. Therefore the only possibly poor vertices are vertices of degree three. If v has degree three and is poor, then it has weight at least six since $ch_0(v) = 11 - 2\mathrm{wt}(v)$. By Lemma 2.4, vertices of degree three (and thus poor vertices) have weight at most eight.

Before proceeding with the analysis, the existence of several types of poor vertices is ruled out. This is accomplished using the characterization of intersecting 7- and 9-cycles, as well as the potentials of subgraphs of G (see Lemma 3.2). In particular, we show that G does not contain vertices of type $(4, 4, 0)$, of type $(4, 3, 1)$, or of type $(3, 3, 2)$. Thus the only poor vertices of weight eight are of type (4,2,2). The poor vertices of weight seven are of type (4,3,0), (4,2,1), (3,3,1), or (3,2,2), and the poor vertices of weight six are of type (4,2,0), (4,1,1), (3,3,0), or (3,2,1).

As not all vertices of degree three and weight at least six can be ruled out outright, much of the remainder of the structural analysis consists of establishing the local structure surrounding the vertices of degree three and the cells and vertices that later send them charge.

We discharge in steps: each step consists of a single rule R that is carried out instantaneously throughout the graph. For convenience, we refer to the rules and steps interchangeably. At the end of Step i., the resulting charge of each cell and vertex is denoted by ch_i. If S is a k-string with $k \leq 2$, we refer to S as a *short* string.

R1. Each vertex contained in a cell sends all of its charge to the cell that contains it. (Since cells are disjoint as shown above, this is unambiguous.)

R2. Let u and v share a short string. If u is in a cell C and v is poor after Step 1, C sends $-ch_1(v)$ charge to v.

R3. Let u and v share a short string with $\deg(u) \geq 4$. If v is poor after Step 2, u sends $-ch_2(v)$ charge to v.

R4. Let u and v share a short string with $\deg(u) = 3$ and $\mathrm{wt}(u) \leq 4$. If v is poor after Step 3, u sends $-ch_3(v)$ charge to v.

R5. Let u and v share a short string with $\deg(u) = 3$ and $\mathrm{wt}(u) = 5$. If v is the only poor vertex that shares a short string with u after Step 4, then u sends $-ch_4(v)$ charge to v.

We note two important facts regarding the discharging rules. First, the rules are performed sequentially. This ensures that in the later steps of the discharging process, we will have uncovered a significant amount of information regarding the local structure of the vertices receiving charge. Second, vertices and cells only send charge along short strings. If a vertex or cell sends charge to many poor structures, it follows that the vertex or cell sending charge has relatively low weight and so consequently has a large amount of charge to spare.

The remainder of the analysis is dedicated to showing that every structure in the graph ultimately has non-negative charge. This contradicts Equation (1), thus ruling out the existence of a minimum counterexample to Theorem 1.4.

4. Open questions

A natural question to wonder is whether or not the density bound obtained in Theorem 1.4 is best possible. We suspect not. Kostochka and Yancey [**9**] showed that if G is k-critical and $k \geq 4$, then $e(G) \geq (\frac{k}{2} - \frac{1}{k-1})v(G) - \frac{k(k-3)}{2(k-1)}$. Later, they showed this is tight for graphs[5] obtained via a construction given by Ore in [**12**]. A k-critical graph given by Ore's construction is called a *k-Ore graph.* Given a $(2t+2)$-critical graph, there is a seemingly natural way to obtain a C_{2t+1}-critical graph by edge subdivisions. Indeed, we have the following.

Proposition 4.1. *If G is a $(2t+2)$-critical graph, then the graph G' obtained from G by subdividing every edge $(2t-2)$ times is C_{2t+1}-critical.*

Since the edge-density obtained by Kostochka and Yancey for k-critical graphs is tight for k-Ore graphs, it seems reasonable that the corresponding density obtained from subdividing a $(2t+2)$-Ore graph could be best possible for C_{2t+1}-critical graphs. This idea motivates the following.

Proposition 4.2. *Let $t \geq 1$ be an integer, and let G be a $(2t+2)$-Ore graph. Let G' be the graph obtained from G by subdividing each edge in $E(G)$ $(2t-2)$ times. Then $e(G') = \frac{t(2t+3)v(G')-(t+1)(2t-1)}{2t^2+2t-1}$.*

We therefore find it reasonable to ask the following question.

Question 4.3. Let $t \geq 3$. Does every C_{2t+1}-critical graph G satisfy $e(G) \geq \frac{t(2t+3)v(G)-(t+1)(2t-1)}{2t^2+2t-1}$?

[5] They showed further that this bound is tight *only* for the graphs obtained via Ore's construction.

We note that the family of graphs described in Proposition 4.2 show that it is impossible to prove Conjecture 1.2 using only a density bound. When $t = 3$, the graphs described in Proposition 4.2 have an asymptotic density of $\frac{27}{23}$. However, using Euler's planar graph formula, we have that if G is a planar graph of girth at least g, then $e(G) \leq \frac{g}{g-2}(v(G)-2)$ —or, asymptotically, that $\frac{e(G)}{v(G)} \leq \frac{g}{g-2}$. In order to obtain a density argument that implies a relaxation of Conjecture 1.2, the girth bound g chosen in the relaxation will satisfy $\frac{g}{g-2} \leq \frac{27}{23}$ —in other words, $g \geq 14$. A proof of Conjecture 1.2 will thus not be a purely density-based argument.

Finally, we note that a positive answer to Question 4.3 together with Euler's formula for planar graphs implies that if G is a planar graph with girth at least $4t+2$, then G admits a homomorphism to C_{2t+1}. The girth bound of $4t+2$ is of particular interest as no counterexamples to the primal version of the conjecture with edge-connectivity $4t+2$ have yet been found.

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