Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 717–722

EDGE ORDERED TURÁN PROBLEMS

D. GERBNER, A. METHUKU, D. T. NAGY, D. PÁLVÖLGYI, G. TARDOS and M. VIZER

Abstract. We introduce the Turán problem for edge ordered graphs. We call a simple graph *edge ordered*, if its edges are linearly ordered. An isomorphism between edge ordered graphs must respect the edge order. A subgraph of an edge ordered graph is itself an edge ordered graph with the induced edge order. We say that an edge ordered graph G *avoids* another edge ordered graph H, if no subgraph of G is isomorphic to H. The Turán number $\mathrm{ex}'_<(n, \mathcal{H})$ of a family $\mathcal{H}$ of edge ordered graphs is the maximum number of edges in an edge ordered graph on n vertices that avoids all elements of $\mathcal{H}$.

We examine this parameter in general and also for several singleton families of edge orders of certain small specific graphs, like star forests, short paths and the cycle of length four.

1. Introduction

The most basic form of a *Turán type extremal problem* asks the maximum number $\mathrm{ex}(n, H)$ of edges in an n-vertex simple graph that does not contain a "forbidden" graph H as a subgraph. For a family $\mathcal{H}$ of forbidden graphs we write $\mathrm{ex}(n, \mathcal{H})$ to denote the maximal number of edges of a simple graph on n vertices that contains no member of $\mathcal{H}$ as a subgraph. This problem has its roots in the works of Mantel [**12**] and Turán [**18**], for recent results consult the survey of Füredi and Simonovits [**6**]. For the extremal theory of graphs with a circular or linear order on their *vertex set*, see Braß, Károlyi and Valtr [**1**] or Tardos [**17**], respectively. Note that vertex

Received June 6, 2019.
2010 *Mathematics Subject Classification.* Primary 05D05, 05C35.
Research of D. Gerbner was supported by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and the National Research, Development and Innovation Office, NKFIH project K 116769.
Research of A. Methuku and D. T. Nagy was supported by the National Research, Development and Innovation Office, NKFIH project K 116769.
Research of D. Pálvölgyi was supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under grant number LP2017-19/2017.
Reseach of G. Tardos was supported by the Cryptography "Lendület" project of the Hungarian Academy of Sciences and by the National Research, Development and Innovation Office, NKFIH projects K 116769 and SSN 117879.
Research of M. Vizer was supported by the National Research, Development and Innovation Office, NKFIH projects SNN 129364 and KH 130371.

ordered graphs are usually called *ordered graphs* in the literature. Here we extend this theory to *edge ordered graphs*.

An *edge ordered graph* is a finite simple graph $G = (V, E)$ with a linear ordering on its edge set E. We often give this linear order with an injective function $L\colon E \to R$, that we call a *labeling*. In this case we denote the edge ordered graph obtained by G^L and we also call it a *labeling* of G.

An isomorphism between edge ordered graphs must respect the edge order. A subgraph of an edge ordered graph is itself an edge ordered graph with the induced edge order. We say that the edge ordered graph G *contains* another edge ordered graph H, if H is isomorphic to a subgraph of G. Otherwise we say that G *avoids* H. We say that G *avoids* a family of edge ordered graphs if it avoids every member of the family. When speaking of a family of edge ordered graphs we always assume that all members of the family are *non-empty*, that is, they have at least one edge. This is necessary for the definition of the Turán number below to make sense. The Turán problem for edge ordered graphs can be formulated as follows.

Definition 1.1. For a positive integer n and a family of edge ordered graphs $\mathcal{H}$ let $\mathrm{ex}'_<(n, \mathcal{H})$, the *Turán number of* $\mathcal{H}$, stand for the maximal number of edges in an edge ordered graph on n vertices that avoids $\mathcal{H}$. In case there is only one forbidden edge ordered graph H we simply write $\mathrm{ex}'_<(n, H)$ to mean $\mathrm{ex}'_<(n, \{H\})$.

Note that Turán problems for vertex ordered graphs (see [**17**]) deal with the same function but with linear ordering on the vertices instead of edges.

We will denote labelings of short paths and cycles by simply giving the labels of the edges along the path or cycle. For example the labeling of P_4 (the path on four vertices) that gives the first edge the label 1, the second edge the label 3, and the last edge the label 2 is denoted by P_4^{132}. Similarly, C_4^{1234} denotes the cyclically increasing labeling of the cycle C_4.

1.1. History

We are only aware of very few particular instances of this problem that have been investigated earlier. In most of these cases one was looking for an increasing path or trail. Call a sequence $v_1, \ldots, v_k$ of vertices in an edge ordered graph an *increasing trail* if $v_i v_{i+1}$ form a strictly increasing sequence of edges for $1 \le i < k$. If all the vertices v_i are distinct, we call it an *increasing path*. Chvátal and Komlós [**3**] proposed to determine the length of the longest increasing trail/path that can be found in all labelings of K_n. Later Graham and Kleitman [**8**] proved that for trails the answer is exactly $n-1$ (if $n \ge 6$) and for paths they obtained the bounds $(\sqrt{4n-3}-1)/2$ from below and $3n/4$ from above. This corresponds to a single forbidden edge ordered graph, namely the increasing path $P_{k+1}^{12\cdots k}$ that we denote by P_{k+1}^{inc}. The question was also studied in arbitrary host graphs where they call the *altitude* of the simple graph G the length of the longest increasing path that can be found in every labeling G^L of G. Note that determining how many edges a simple graph with a given altitude and number of vertices can have is equivalent to finding $\mathrm{ex}'_<(n, P_k^{\mathrm{inc}})$. For more recent results on this problem, see e.g. [**2, 14**] and the references therein.

Rödl [**15**] proved that any graph with average degree at least $k(k+1)$ has altitude at least k. In our notation this can be formulated as $\mathrm{ex}'_<(n, P_k^{\mathrm{inc}}) < \binom{k}{2}n$. In terms of k, this is far from the lower bound $\mathrm{ex}'_<(n, P_k^{\mathrm{inc}}) \geq \mathrm{ex}(n, P_k) = \frac{k-2}{2}n - O(k^2)$.

A result of Tardos [**16**] implies $\mathrm{ex}'_<(n, \mathcal{T}_5^{1432}) = O(n \log n)$, where $\mathcal{T}_5^{1432}$ denotes the family of trails corresponding to the edge ordered path P_5^{1432} (that is, $\mathcal{T}_5^{1432}$ consists of P_5^{1432} and the three edge ordered graphs obtained from it by identifying vertices of distance at least three).

The only result we are aware of where the forbidden edge ordered graph is neither a path, nor a trail, is due to Gerbner, Patkós and Vizer [**7**]. They proved $\mathrm{ex}'_<(n, C_4^{1243}) = O(n^{5/3})$. This result was the starting point of our research.

2. General results

The most general result in Turán-type extremal graph theory is the Erdős-Stone-Simonovits theorem:

Theorem 2.1 (Erdős-Stone-Simonovits theorem [**4, 5**])**.** *Let $\mathcal{H}$ be a family of simple graphs and $r + 1 = \min\{\chi(H) : H \in \mathcal{H}\} \geq 2$. We have*

$$\mathrm{ex}(n, \mathcal{H}) = \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}.$$

Here $\chi(H)$ stands for the *chromatic number* of the graph H. The key to extend this result to edge ordered graphs is to find the notion that can play the role of the chromatic number in the original theorem. We do this as follows.

Definition 2.2. We say that a simple graph G *can avoid* a family $\mathcal{H}$ of edge ordered graphs, if there is a labeling G^L of G that avoids all members of $\mathcal{H}$.

Let $\chi'_<(\mathcal{H})$, the *order chromatic number* of $\mathcal{H}$, stand for the smallest chromatic number $\chi(G)$ of a finite graph G that cannot avoid $\mathcal{H}$. In case all finite simple graphs can avoid $\mathcal{H}$ we define $\chi'_<(\mathcal{H}) = \infty$. In case the family $\mathcal{H}$ contains a single edge ordered graph we write $\chi'_<(H)$ to denote $\chi'_<(\{H\})$.

Remark. Recall that when speaking of a family of edge ordered graphs we assume no member of the family is empty. This makes the order chromatic number at least 2.

Theorem 2.3 (Erdős-Stone-Simonovits theorem for edge ordered graphs)**.**
If $\chi'_<(\mathcal{H}) = \infty$, then

$$\mathrm{ex}'_<(n, \mathcal{H}) = \binom{n}{2}.$$

If $\chi'_<(\mathcal{H}) = r + 1 < \infty$, then

$$\mathrm{ex}'_<(n, \mathcal{H}) = \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}.$$

Whereas the asymptotics of the Turán number of a family of simple graphs depends on the lowest chromatic number of a *single graph* in the family, in our results the order chromatic number of the *entire family* counts. This is a meaningful difference for order chromatic numbers 3 and up, but not for 2 as we could prove the following.

Theorem 2.4. *We have*

- $\chi'_<(\mathcal{H}) = 2$ *if and only if there exists* $G \in \mathcal{H}$ *with* $\chi'_<(G) = 2$, *and*
- $\chi'_<(P_5^{1423}) = \chi'_<(P_5^{2314}) = \infty$, *but* $\chi'_<(\{P_5^{1423}, P_5^{2314}\}) = 3$.

2.1. Results about the best and worst orderings of a graph

As different labelings of the same underlying graph can have really different order chromatic numbers, it is natural to investigate the following: a non-empty finite graph G, let $\chi^-_<(G) := \min_L \chi'_<(G^L)$, where the minimum is over all labelings L of G. Similarly, let $\chi^+_<(G) := \max_L \chi'_<(G^L)$. We can determine $\chi^-_<(G)$ for many simple graphs G and $\chi^+_<(G)$ for all of them.

Proposition 2.5. *We have:*

- $\chi^-_<(G) \geq \chi(G)$ *for any graph* G, *and*
- $\chi^-_<(G) = 2$ *if and only if* $\chi(G) = 2$.

Proposition 2.6. $\chi^-_<(K_4) = \infty$.

Recall that a *star* is a simple, connected graph in which all edges share a common vertex and a *star forest* is a non-empty graph whose connected components are all stars. We allow isolated vertices (as empty stars) but require that a star forest is not empty.

Theorem 2.7. *We have* $\chi^+_<(G) = 2$ *if* G *is a star forest or* $G = P_4$. *We have* $\chi^+_<(K_3) = 3$. *For all remaining non-empty simple graphs* G *we have* $\chi^+_<(G) = \infty$.

3. Results about specific edge ordered graphs

3.1. Star forests

We could connect the Turán function of edge ordered star forests to Davenport-Schinzel theory (for an introduction see e.g., [**11**]) and prove the following:

Theorem 3.1. *Let* F *be an edge ordered star forest and let* $\alpha(n)$ *be the inverse Ackermann function. We have*

$$\mathrm{ex}'_<(n, F) \leq n2^{\alpha(n)^c},$$

where the exponent c *depends on* F, *but not on* n.

Let F_0 *be the edge ordered star forest, with five edges such that the first, third and fifth edges form a star component and the second and fourth edges form another component. We have*

$$\mathrm{ex}'_<(n, F_0) = \Omega(n\alpha(n)).$$

Theorem 3.2. *For edge ordered star forests* F *on at most 4 edges we have*

$$\mathrm{ex}'_<(n, F) = O(n).$$

3.2. Paths

As we have mentioned in the introduction, known results on the altitude of graphs imply a linear upper bound on the number of edges if the monotone labeling of the path P_k is forbidden. We can also estimate the Turán numbers of other labelings of paths on four or five vertices.

Theorem 3.3. *For an edge ordered path P_4^L on four vertices we have*

$$\mathrm{ex}'_<(n, P_4^L) = \Theta(n).$$

The labelings of P_5 are given by permutations of $\{1, 2, 3, 4\}$. Labelings obtained from one another by reversing the path or reversing the edge order yield equal Turán numbers. This makes for eight equivalence classes of the labelings of P_5. We could determine the order of magnitude of the edge ordered Turán number for all but one of them.

Theorem 3.4.

- *If* $L \in \{1234, 4321\} \cup \{1243, 3421, 4312, 2134\}$, *then we have* $\mathrm{ex}'_<(n, P_5^L) = \Theta(n)$.
- *If* $L \in \{1324, 4231\} \cup \{1432, 2341, 4123, 3214\} \cup \{2143, 3412\}$, *then we have* $\mathrm{ex}'_<(n, P_5^L) = \Theta(n \log n)$.
- *If* $L \in \{2413, 3142\} \cup \{1423, 3241, 4132, 2314\}$, *then we have* $\mathrm{ex}'_<(n, P_5^L) = \binom{n}{2}$.
- *If* $L \in \{1342, 2431, 4213, 3124\}$, *then we have* $\mathrm{ex}'_<(n, P_5^L) = \Omega(n \log n)$ *and* $\mathrm{ex}'_<(n, P_5^L) = O(n \log^2 n)$

3.3. The 4-cycles

The automorphisms of C_4 leave only three non-isomorphic labelings of C_4. It is easy to see that $\mathrm{ex}'_<(n, C_4^{1234}) = \mathrm{ex}'_<(n, C_4^{1324}) = \binom{n}{2}$. Concerning the third possible labeling of C_4 we improve the upper bound $\mathrm{ex}'_<(n, C_4^{1243}) = O(n^{5/3})$ that appeared in [**7**]. Our new bound is close to the trivial lower bound of $\mathrm{ex}'_<(n, C_4^{1243}) \geq \mathrm{ex}(n, C_4) = \Theta(n^{3/2})$.

Theorem 3.5. $\mathrm{ex}'_<(n, C_4^{1243}) = O(n^{3/2} \log n)$.

The proof of this theorem is inspired by [**13**].

References

1. Braß P., Károlyi Gy. and Valtr P., *A Turán-type extremal theory of convex geometric graphs*, in: Discrete and Computational Geometry, Springer, Berlin, Heidelberg, 2003, 275–300.
2. Bucić M., Kwan M., Pokrovskiy A., Sudakov B., Tran T. and Wagner A. Z., *Nearly-linear monotone paths in edge-ordered graphs*, `arXiv:1809.01468`.
3. Chvátal V. and Komlós J., *Some combinatorial theorems on monotonicity*, Canad. Math. Bull. **14** (1971), 1–3.
4. Erdős P. and Simonovits M., *A limit theorem in graph theory*, Studia Sci. Math. Hungar. **1** (1966), 51–57.

5. Erdős P. and Stone A. H., *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091.
6. Füredi Z. and Simonovits M., *The history of degenerate (bipartite) extremal graph problems*, in: Erdős Centennial, Springer, 2013, 169–264.
7. Gerbner D., Patkós B. and Vizer M., *Forbidden subposet problems for traces of set families*, Electron. J. Combin. **25** (2018), #3.49.
8. Graham R. L. and Kleitman D. J., *Increasing paths in edge ordered graphs*, Period. Math. Hungar. **3** (1973), 141–148.
9. Hart S. and Sharir M., *Nonlinearity of Davenport-Schinzel sequences and of generalized path compression schemes*, Combinatorica **6** (1986), 151–177.
10. Keevash P., *Hypergraph Turán problems*, Surveys in Combinatorics **392** (2011), 83–140.
11. Klazar M., *A general upper bound in extremal theory of sequences*, Comment. Math. Univ. Carolin. **33**, (1992) 737–746.
12. Mantel W., *Problem 28 (Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff)*, Wiskundige Opgaven **10** (1907), 60–61.
13. Marcus A. and Tardos G., *Intersection reverse sequences and geometric applications*, J. Combin. Theory, Ser. A **113** (2006), 675–691.
14. Milans K. G., *Monotone paths in dense edge-ordered graphs*, J. Combin. **8** (2017), 423–437.
15. Rödl V., Master's thesis, Charles University, 1973.
16. Tardos G., *Construction of locally plane graphs with many edges*, in: Thirty Essays on Geometric Graph Theory, Springer, New York, NY, 2013, 541–562.
17. Tardos G., *Extremal theory of ordered graphs*, in: Proceedings of the International Congress of Mathematics – 2018, **3** (2018), 3219–3228.
18. Turán P., *On an extremal problem in graph theory* , Matematikai és Fizikai Lapok, **48** (1941), 436–452, (in Hungarian).

D. Gerbner, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary,
e-mail: `gerbner@renyi.hu`

A. Methuku, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland and Central European University, Budapest, Hungary,
e-mail: `abhishekmethuku@gmail.com`

D. T. Nagy, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary,
e-mail: `nagydani@renyi.hu`

D. Pálvölgyi, MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics, Eötvös Loránd University (ELTE), Budapest, Hungary

G. Tardos , Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary,
e-mail: `tardos@renyi.hu`

M. Vizer, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary,
e-mail: `vizermate@gmail.com`