EDGE-ORDERED RAMSEY NUMBERS

M. BALKO AND M. VIZER

Abstract. We introduce and study a variant of Ramsey numbers for edge-ordered graphs, that is, graphs with linearly ordered sets of edges. The edge-ordered Ramsey number \( R_e(G) \) of an edge-ordered graph \( G \) is the minimum positive integer \( N \) such that there exists an edge-ordered complete graph \( K_N \) on \( N \) vertices such that every 2-coloring of the edges of \( K_N \) contains a monochromatic copy of \( G \) as an edge-ordered subgraph of \( K_N \).

We prove that the edge-ordered Ramsey number \( R_e(G) \) is finite for every edge-ordered graph \( G \) and we obtain better estimates for special classes of edge-ordered graphs. In particular, we prove \( R_e(G) \leq 2O(n^3 \log n) \) for every bipartite edge-ordered graph \( G \) on \( n \) vertices. We also introduce a natural class of edge-orderings, called lexicographic edge-orderings, for which we can prove much better upper bounds on the corresponding edge-ordered Ramsey numbers.

1. Introduction

An edge-ordered graph \( \mathcal{G} = (G, \prec) \) consists of a graph \( G = (V, E) \) and a linear ordering \( \prec \) of the set of edges \( E \). We sometimes use the term edge-ordering of \( G \) for the ordering \( \prec \) and also for \( \mathcal{G} \). An edge-ordered graph \( (G, \prec_1) \) is an edge-ordered subgraph of an edge-ordered graph \( (H, \prec_2) \) if \( G \) is a subgraph of \( H \) and \( \prec_1 \) is a suborder of \( \prec_2 \). We say that \( (G, \prec_1) \) and \( (H, \prec_2) \) are isomorphic if there is a graph isomorphism between \( G \) and \( H \) that also preserves the edge-orderings \( \prec_1 \) and \( \prec_2 \).

For a positive integer \( k \), a \( k \)-coloring of the edges of a graph \( G \) is any function that assigns one of the \( k \) colors to each edge of \( G \). The edge-ordered Ramsey number \( R_e(\mathcal{G}) \) of an edge-ordered graph \( \mathcal{G} \) is the minimum positive integer \( N \) such that there exists an edge- ordering \( \mathcal{R}_N \) of the complete graph \( K_N \) on \( N \) vertices such that every 2-coloring of the edges of \( \mathcal{R}_N \) contains a monochromatic copy of \( \mathcal{G} \) as an edge-ordered subgraph of \( \mathcal{R}_N \). More generally, for two edge-ordered graphs \( \mathcal{G} \) and \( \mathcal{H} \), we use \( R_e(\mathcal{G}, \mathcal{H}) \) to denote the minimum positive integer \( N \) such that there exists an edge-ordering \( \mathcal{R}_N \) of \( K_N \) such that every 2-coloring of the edges of \( \mathcal{R}_N \) contains a monochromatic copy of \( \mathcal{G} \) as an edge-ordered subgraph of \( \mathcal{R}_N \).

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K with colors red and blue contains a red copy of $G$ or a blue copy of $H$ as an edge-ordered subgraph of $K_N$.

To our knowledge, Ramsey numbers of edge-ordered graphs were not considered in the literature. On the other hand, Ramsey numbers of graphs with ordered vertex sets have been quite extensively studied recently; for example, see [2, 3, 6]. For questions concerning extremal problems about vertex-ordered graphs consult the recent surveys [14, 15]. A vertex-ordered graph $G = (G, \prec)$ (or simply an ordered graph) is a graph $G$ with a fixed linear ordering $\prec$ of its vertices. We use the term vertex-ordering of $G$ to denote the ordering $\prec$ as well as the ordered graph $G$. An ordered graph $(G, \prec_1)$ is a vertex-ordered subgraph of an ordered graph $(H, \prec_2)$ if $G$ is a subgraph of $H$ and $\prec_1$ is a suborder of $\prec_2$. We say that $(G, \prec_1)$ and $(H, \prec_2)$ are isomorphic if there is a graph isomorphism between $G$ and $H$ that also preserves the vertex-orderings $\prec_1$ and $\prec_2$. Unlike in the case of edge-ordered graphs, there is a unique vertex-ordering $K_N$ of $K_N$ up to isomorphism. The ordered Ramsey number $R(G)$ of an ordered graph $G$ is the minimum $N \in \mathbb{N}$ such that every 2-coloring of the edges of $K_N$ contains a monochromatic copy of $G$ as a vertex-ordered subgraph of $K_N$.

For an $n$-vertex graph $G$, let $R(G)$ be the Ramsey number of $G$. It is easy to see that $R(G) \leq \overline{R}(G)$ and $R(G) \leq \overline{R}_v(\emptyset)$ for each vertex-ordering $G$ of $G$ and edge-ordering $\emptyset$ of $G$. We also have $\overline{R}(G) \leq \overline{R}(K_n) = R(K_n)$ and thus ordered Ramsey numbers are always finite. Proving that $\overline{R}_v(\emptyset)$ is always finite seems to be more challenging; see Theorem 2.1.

The Turán numbers of edge-ordered graphs were recently introduced in [7]. The authors of [7] proved, for example, a variant of the Erdős–Stone–Simonovits Theorem for edge-ordered graphs, and also investigated the Turán numbers of small edge-ordered paths, star forests, and 4-cycles; see the last section of [15].

2. Our results

We study the growth rate of edge-ordered Ramsey numbers with respect to the number of vertices for various classes of edge-ordered graphs. As our first result, we show that edge-ordered Ramsey numbers are always finite and thus well-defined.

**Theorem 2.1.** For every edge-ordered graph $\emptyset$, the number $\overline{R}_v(\emptyset)$ is finite.

Theorem 2.1 also follows from a recent deep result of Hubička and Nešetřil [9, Theorem 4.33] about Ramsey numbers of general relational structures. In comparison, our proof of Theorem 2.1 is less general, but it is much simpler and produces better and more explicit bound on $\overline{R}_v(\emptyset)$. It is a modification of the proof of Theorem 12.13 [12, Page 138], which is based on the Graham–Rothschild Theorem [8]. In fact, the proof of Theorem 2.1 yields a stronger induced-type statement where additionally the ordering of the vertex set is fixed. Theorem 2.1 can also be extended to $k$-colorings with $k > 2$.

Due to the use of the Graham–Rothschild Theorem, the bound on the edge-ordered Ramsey numbers obtained in the proof of Theorem 2.1 is still enormous. It follows from a result of Shelah [13, Theorem 2.2] that this bound on $\overline{R}_v(\emptyset)$ is
primitive recursive, but it grows faster than, for example, a tower function of any fixed height. Thus we aim to prove more reasonable estimates on edge-ordered Ramsey numbers, at least for some classes of edge-ordered graphs.

As our second main result, we show that one can obtain a much better upper bound on edge-ordered Ramsey numbers of two edge-ordered graphs, provided that one of them is bipartite. For $d \in \mathbb{N}$, we say that a graph $G$ is $d$-degenerate if every subgraph of $G$ has a vertex of degree at most $d$.

**Theorem 2.2.** Let $\mathcal{G}$ be a $d$-degenerate edge-ordered graph on $n'$ vertices and let $\mathcal{G}$ be a bipartite edge-ordered graph with $m$ edges and with both parts containing $n$ vertices. If $d \leq n$ and $n' \leq t^{d+1}$ for $t = 3n^{10}m!$, then

$$R_e(\mathcal{G}, \mathcal{G}) \leq (n')^2 t^{d+1}.$$

In particular, if $\mathcal{G}$ is a bipartite edge-ordered graph on $n$ vertices, then $R_e(\mathcal{G}) \leq 2^{O(n^2 \log n)}$. We believe that the bound can be improved. In fact, it is possible that $R_e(\mathcal{G})$ is at most exponential in the number of vertices of $\mathcal{G}$ for every edge-ordered graph $\mathcal{G}$. We note that, for every graph $G$ and its vertex-ordering $\sigma$, both the standard Ramsey number $R(G)$ and the ordered Ramsey number $R_e(G)$ grow at most exponentially in the number of vertices of $G$.

In general, the difference between edge-ordered Ramsey numbers and ordered Ramsey numbers with the same underlying graph can be very large. Let $M_n$ be a matching on $n$ vertices, that is, a graph formed by a collection of $n/2$ disjoint edges. There are ordered matchings $M_n = (M_n, <)$ with super-polynomial ordered Ramsey numbers $R_e(M_n)$ in $n$ [2, 6]. In fact this is true for almost all ordered matchings on $n$ vertices [6]. On the other hand, all edge-orderings of $M_n$ are isomorphic as edge-ordered graphs and thus $R_e(M_n) = R(M_n) = O(n)$ for every edge-ordering $\mathcal{M}_n$ of $M_n$.

We consider a special class of edge-orderings, which we call **lexicographic edge-orderings**, for which we can prove much better upper bounds on their edge-ordered Ramsey numbers and which seem to be quite natural.

An ordering $\prec$ of edges of a graph $G = (V, E)$ is **lexicographic** if there is a one-to-one correspondence $f : V \rightarrow \{1, \ldots, |V|\}$ such that any two edges $\{u, v\}$ and $\{w, t\}$ of $G$ with $f(u) < f(v)$ and $f(w) < f(t)$ satisfy $\{u, v\} \prec \{w, t\}$ if either $f(u) < f(w)$ or if $f(u) = f(w)$ and $f(v) < f(t)$. We say that such mapping $f$ is **consistent** with $\prec$. Note that, for every vertex $u$, the edges $\{u, v\}$ with $f(u) < f(v)$ form an interval in $\prec$. Also observe that there is a unique (up to isomorphism) lexicographic edge-ordering $\mathcal{G}_n^{\text{lex}}$ of $K_n$. Setting $\{u, v\} \prec' \{w, t\}$ if either $f(u) < f(w)$ or if $f(u) = f(w)$ and $f(v) > f(t)$ we obtain the **max-lexicographic** edge-ordering $\prec'$ of $G$.

For a linear ordering $\prec$ on some set $X$, we use $\prec^{-1}$ to denote the inverse ordering of $\prec$, that is, for all $x, y \in X$, we have $x \prec^{-1} y$ if and only if $y \prec x$.

The lexicographic and max-lexicographic edge-orderings are natural, as Nešetřil and Rödl [11] showed that these orderings are canonical in the following sense.

**Theorem 2.3 ([11]).** For every $n \in \mathbb{N}$, there is a positive integer $T(n)$ such that every edge-ordered complete graph on $T(n)$ vertices contains a copy of $K_n$ such that...
the edges of this copy induce one of the following four edge-orderings: lexicographic edge-ordering $\prec$, max-lexicographic edge-ordering $\prec'$, $\prec^{-1}$, or $(\prec')^{-1}$.

Theorem 2.3 is also an unpublished result of Leeb; see [10]. It is thus natural to consider the following variant of edge-ordered Ramsey numbers, which turns out to be more tractable than general edge-ordered Ramsey numbers. The lexicographic edge-ordered Ramsey number $\overline{R}_{\text{lex}}(\mathcal{G})$ of a lexicographically edge-ordered graph $\mathcal{G}$ is the minimum $N$ such that every 2-coloring of the edges of $\mathcal{G}$ contains a monochromatic copy of $\mathcal{G}$ as an edge-ordered subgraph of $\mathcal{G}_N$. Observe that $\overline{R}_{\text{lex}}(\mathcal{G}) \leq \overline{R}_{\text{lex}}(\mathcal{H})$ for every lexicographically edge-ordered graph $\mathcal{H}$.

For every lexicographically edge-ordered graph $\mathcal{G} = (G, \prec)$, the lexicographic edge-ordered Ramsey number $\overline{R}_{\text{lex}}(\mathcal{G})$ can be estimated from above with the ordered Ramsey number of some vertex-ordering of $G$. More specifically,

$$\overline{R}_{\text{lex}}(\mathcal{G}) \leq \min_f \overline{R}(G_f),$$

where the minimum is taken over all one-to-one correspondences $f : V \rightarrow \{1, \ldots, |V|\}$ that are consistent with the lexicographic edge-ordering $\mathcal{G}$ and $G_f$ is the vertex-ordering of $G$ determined by $f$. Since $\overline{R}(K_n) = R(K_n)$, it follows from (1) and from the well-known bound $R(K_n) \leq 2^{2n}$ that the numbers $\overline{R}_{\text{lex}}(\mathcal{G})$ are always at most exponential in the number of vertices of $G$. In fact, we have $\overline{R}_{\text{lex}}(\mathcal{G}_{n}) = \overline{R}(K_n) = R(K_n)$ for every $n$. The equality is achieved in (1), for example, for graphs with a unique vertex-ordering determined by the lexicographic edge-ordering. Such graphs include graphs where each edge is contained in a triangle. Additionally, combining (1) with a result of Conlon et al. [6, Theorem 3.6] gives the estimate

$$\overline{R}_{\text{lex}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}$$

for every $d$-degenerate lexicographically edge-ordered graph $\mathcal{G}$ on $n$ vertices. In particular, $\overline{R}_{\text{lex}}(\mathcal{G})$ is at most quasi-polynomial in $n$ if $d$ is fixed.

We note that the bound (1) is not always tight. For example, $R(K_{1,n}) = \overline{R}_{\text{lex}}(\mathcal{G}_{1,n})$ for every edge-ordering $\mathcal{G}_{1,n}$ of $K_{1,n}$, as any two edge-ordered stars $K_{1,n}$ are isomorphic as edge-ordered graphs. However, the Ramsey number $R(K_{1,n})$ is known to be strictly smaller than $\overline{R}(K_{1,n})$ for $n$ even and for any vertex-ordering $K_{1,n}$ of $K_{1,n}$; see [4] and [1, Observation 11 and Theorem 12].

Using the inequality (1) we obtain asymptotically tight estimate on the following lexicographic edge-ordered Ramsey numbers of paths. The edge-monotone path $\mathcal{P}_n = (P_n, \prec)$ is the edge-ordered path on vertices $v_1, \ldots, v_n$, where $\{v_1, v_2\} \prec \cdots \prec \{v_{n-1}, v_n\}$.

**Proposition 2.4.** For every integer $n > 2$, we have $\overline{R}_{\text{lex}}(\mathcal{P}_n) \leq 2n - 3 + \sqrt{2n^2 - 8n + 11}$.

The proof of Proposition 2.4 uses the fact that the one-to-one correspondence $f$ consistent with the lexicographic edge-ordering of $P_n$ is not determined uniquely. Indeed, we can choose the mapping $f$ so that it determines the vertex-ordering $P_n$.
of $P_n$ where edges are between consecutive pairs of vertices. Such vertex-ordering $P_n$ is called monotone path. However, it is known that $R(P_n) = (n-1)^2 + 1$ [5] and thus we cannot apply (1) to this ordering to obtain a linear bound on $R_{\text{lex}}(P_n)$. Instead we choose a different mapping $f$ that determines a vertex-ordering of $P_n$ with linear ordered Ramsey number.

Finally, we show an upper bound on edge-ordered Ramsey numbers of two graphs, where one of them is bipartite and suitably lexicographically edge-ordered. This result uses a stronger assumption about $\mathcal{G}$ than Theorem 2.2, but gives much better estimate. For $m, n \in \mathbb{N}$, let $K_{m,n}^{\text{lex}}$ be the lexicographic edge-ordering of $K_{m,n}$ that induces a vertex-ordering, in which both parts of $K_{m,n}$ form an interval.

**Theorem 2.5.** Let $H$ be a $d$-degenerate edge-ordered graph on $n'$ vertices and let $G$ be an edge-ordered subgraph of $K_{m,n}^{\text{lex}}$. Then

$$R_e(H, G) \leq (n')^2 n'^{d+1}.$$ 

3. Open problems

Many questions about edge-ordered Ramsey numbers remain open, for example proving a better upper bound on edge-ordered Ramsey numbers than the one obtained in the proof of Theorem 2.1. For general upper bounds, it suffices to focus on edge-ordered complete graphs. It is possible that edge-ordered Ramsey numbers of edge-ordered complete graphs do not grow significantly faster than the standard Ramsey numbers.

**Problem 3.1.** Is there a constant $C$ such that, for every $n \in \mathbb{N}$ and every edge-ordered complete graph $\mathcal{K}_n$ on $n$ vertices, we have $R_e(\mathcal{K}_n) \leq 2^{O(n)}$?

It might also be interesting to consider sparser graphs and try to prove better upper bounds on their edge-ordered Ramsey numbers.

Another interesting open problem is to determine the growth rate of the function $T(n)$ from Theorem 2.3. The current upper bound on $T(n)$ is quite large as the proof of Nešetřil and Rödl [11] uses Ramsey’s theorem for quadruples and $6! = 720$ colors.

**References**


M. Balko, Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Czech Republic,
e-mail: balko@kam.mff.cuni.cz

M. Vizer, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary,
e-mail: vizermate@gmail.com