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GENERALIZED TURÁN PROBLEMS FOR EVEN CYCLES

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ABSTRACT. Given a graph H and a set of graphs $\mathcal{F}$, let $\text{ex}(n, H, \mathcal{F})$ denote the maximum possible number of copies of H in an $\mathcal{F}$-free graph on n vertices. We investigate the function $\text{ex}(n, H, \mathcal{F})$, when H and members of $\mathcal{F}$ are cycles. Let C_k denote the cycle of length k and let $\mathscr{C}_k = \{C_3, C_4, \ldots, C_k\}$. We highlight the main results below.

(i) We show that $\text{ex}(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $l, k \geq 2$. Moreover, in some cases we determine it asymptotically.

(ii) Erdős's Girth Conjecture states that for any positive integer k, there exist a constant $c > 0$ depending only on k, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ with girth more than $2k$.

Solymosi and Wong proved that if this conjecture holds, then for any $l \geq 3$ we have $ex(n, C_{2l}, \mathscr{C}_{2l-1}) = \Theta(n^{2l/(l-1)})$. We prove that their result is sharp in the sense that forbidding any other even cycle decreases the number of C_{2l}'s significantly.

(iii) We prove $\text{ex}(n, C_{2l+1}, \mathscr{C}_{2l}) = \Theta(n^{2+1/l})$, provided a stronger version of Erdős's Girth Conjecture holds (which is known to be true when $l = 2, 3, 5$). This result is also sharp in the sense that forbidding one more cycle decreases the number of C_{2l+1}'s significantly.

1. INTRODUCTION

The Turán problem for a set of graphs $\mathcal{F}$ asks the following. What is the maximum number $\text{ex}(n, \mathcal{F})$ of edges that a graph on n vertices can have without containing any $F \in \mathcal{F}$ as a subgraph? When $\mathcal{F}$ contains a single graph F, we simply write $\text{ex}(n, F)$. This function has been intensively studied, starting with Mantel [**17**] who determined $\text{ex}(n, K_3)$ and with Turán [**21**] who determined $\text{ex}(n, K_r)$ for every r, where K_r denotes the complete graph on r vertices with $r \geq 3$. See [**9**] for surveys on this topic.

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For some integer k let C_k denote a cycle on k vertices and let $\mathscr{C}_k$ denote the set $\{C_3, C_4, \ldots, C_k\}$. For even cycles C_{2k}, Bondy and Simonovits [**4**] proved the following upper bound.

Theorem 1 (Bondy, Simonovits [**4**])**.** *For $k \geq 2$ we have*

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k}).$$

The order of magnitude in the above theorem is known to be sharp only for $k = 2, 3, 5$. If all the cycles in $\mathscr{C}_k$ are forbidden, then Alon, Hoory and Linial [**1**] proved the following.

Theorem 2 (Alon, Hoory, Linial [**1**])**.** *For any $k \geq 2$ we have*

(i) $\mathrm{ex}(n, \mathscr{C}_{2k}) < \frac{1}{2}n^{1+1/k} + \frac{1}{2}n$,
(ii) $\mathrm{ex}(n, \mathscr{C}_{2k+1}) < \frac{1}{2^{1+1/k}}n^{1+1/k} + \frac{1}{2}n$.

For more information on the Turán number of cycles one can consult the survey [**22**].

1.1. Generalized Turán problems

For two graphs H and G, let $\mathcal{N}(H, G)$ denote the number of copies of H in G. Given a graph H and a set of graphs $\mathcal{F}$, let

$$\mathrm{ex}(n, H, \mathcal{F}) = \max_G\{\mathcal{N}(H, G) : G \text{ is an } \mathcal{F}\text{-free graph on } n \text{ vertices.}\}$$

If $\mathcal{F} = \{F\}$, we simply denote it by $\mathrm{ex}(n, H, F)$. This problem was initiated by Erdős [**6**], who determined $\mathrm{ex}(n, K_s, K_t)$ exactly. Concerning cycles, Bollobás and Győri [**3**] proved that

$$(1 + o(1))\frac{1}{3\sqrt{3}}n^{3/2} \leq \mathrm{ex}(n, C_3, C_5) \leq (1 + o(1))\frac{5}{4}n^{3/2}$$

and this result was extended by Győri and Li [**14**] for $\mathrm{ex}(n, C_3, C_{2k+1})$ $(k > 2)$. Other improvements can be found in [**8**].

Another notable result is to determine the value of $\mathrm{ex}(n, C_5, C_3)$ by Hatami, Hladký, Král, Norine, and Razborov [**15**] and independently by Grzesik [**12**], where they showed that it is equal to $(\frac{n}{5})^5$. Very recently, the asymptotic value of $\mathrm{ex}(n, C_k, C_{k-2})$ was determined for every odd k by Grzesik and Kielak in [**13**].

1.2. Forbidding a set of cycles

The famous Girth Conjecture of Erdős [**5**] asserts the following.

Conjecture 3 (Erdős's Girth Conjecture [**5**])**.** For any positive integer k, there exist a constant $c > 0$ depending only on k, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ and the girth of G_n is more than $2k$.

This conjecture has been verified for $k = 2, 3, 5$, see [**2, 23**]. For a general k, Sudakov and Verstraëte [**20**] showed that if such graphs exist, then they contain a C_{2l} for any l with $k < l \leq Cn$, for some constant $C > 0$. More recently, Solymosi and Wong [**19**] proved that if such graphs exist, then in fact, they contain many C_{2l}'s for any fixed $l > k$. More precisely they proved:

Theorem 4 (Solymosi, Wong [**19**])**.** *If Erdős's Girth Conjecture holds for k, then for every $l > k$ we have*

$$\mathrm{ex}(n, C_{2l}, \mathscr{C}_{2k}) = \Omega(n^{2l/k}).$$

Remark 1. It is easy to see that if $k+1$ divides $2l$, then $\mathrm{ex}(n, C_{2l}, \mathscr{C}_{2k}) = O(n^{2l/k})$. Indeed, let us associate to each C_{2l}, one fixed ordered list of $2l/(k+1)$ edges $(e_1, e_{k+1}, e_{2k+1}, \ldots)$, where e_1 appears as the first edge (chosen arbitrarily) on the C_{2l}, e_{k+1} as the $(k+1)$-th edge, e_{2k+1} as the $(2k+1)$-th edge and so on. Note that at most one C_{2l} is associated to an ordered tuple $(e_1, e_{k+1}, e_{2k+1}, \ldots)$, because there is at most one path of length $k-1$ connecting the endpoints of any two edges (as all the short cycles are forbidden). Since there are at most $O(n^{1+1/k})$ ways to select each edge, this shows the number of C_{2l}'s is at most $O((n^{1+1/k})^{2l/(k+1)}) = O(n^{2l/k})$, showing that the bound in Theorem 4 is sharp when $k+1$ divides $2l$.

2. Our results

Note that all the proofs of the results (and even more results) can be found in [**10**], the article version of this extended abstract. For any two positive integers n and l, let $(n)_l$ denote the product $n(n-1)(n-2)\ldots(n-(l-1))$.

2.1. Forbidding a cycle of given length

We determine the order of magnitude of $\mathrm{ex}(n, C_{2l}, C_{2k})$ below.

Theorem 5.

- *For any $l \geq 3$ and $k \geq 2$ we have* $\mathrm{ex}(n, C_{2l}, C_{2k}) \leq (1+o(1))\frac{2^{l-2}(k-1)^l}{2l}n^l$.
- *For any $k > l \geq 2$ we have* $\mathrm{ex}(n, C_{2l}, C_{2k}) \geq (1+o(1))\frac{(k-1)_l}{2l}n^l$.
- *For any $l > k \geq 3$ we have* $\mathrm{ex}(n, C_{2l}, C_{2k}) \geq (1+o(1))\frac{1}{l^l}n^l$.

Theorem 5 and Theorem 6 (stated below) show that $\mathrm{ex}(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $k, l \geq 2$, except for the lower bound in the case $k=2$, which can be easily shown by counting cycles in the orthogonal polarity graph of the classical projective plane constructed by Erdős and Rényi [**7**].

We note that Theorem 5 has been proven independently by Gishboliner and Shapira [**11**] and recently extended by Morrison, Roberts and Scott in [**18**].

Solymosi and Wong [**19**] asked whether a similar lower bound (to that of Theorem 4) on the number of C_{2l}'s holds, if just C_{2k} is forbidden instead of forbidding $\mathscr{C}_{2k}$. Theorem 5 answers this question in the negative.

Asymptotic results. We determine $\mathrm{ex}(n, C_4, C_{2k})$ asymptotically.

Theorem 6. *For $k \geq 2$ we have*

$$\mathrm{ex}(n, C_4, C_{2k}) = (1+o(1))\frac{(k-1)(k-2)}{4}n^2.$$

In these theorems most constructions are bipartite, so it is natural to consider the bipartite version of the generalized Turán function: Let $\mathrm{ex}_{\mathrm{bip}}(n, C_{2l}, C_{2k})$ denote the maximum number of copies of a C_{2l} in a bipartite C_{2k}-free graph on n

vertices. Our methods give sharper bounds for $\mathrm{ex}_{\mathrm{bip}}(n, C_{2l}, C_{2k})$ compared to the bounds in Theorem 5 and in the case $l = 3,\ k = 4$ we can even determine the asymptotics.

Theorem 7. *We have*

$$\mathrm{ex}_{\mathrm{bip}}(n, C_6, C_8) = n^3 + O(n^{5/2}).$$

2.2. Forbidding a set of cycles

It is easy to see that when counting copies of an even cycle, forbidding an odd cycle does not change the order of magnitude. Therefore by Theorem 4 and Remark 1 we have

Corollary 8. *Suppose $l \geq 3$ and Erdős's Girth Conjecture is true for $l - 1$. Then we have*

$$\mathrm{ex}(n, C_{2l}, \mathscr{C}_{2l-1}) = \Theta(n^{2l/(l-1)}).$$

So the maximum number of C_{2l}'s in a graph of girth $2l$ is $\Theta(n^{2l/(l-1)})$. We prove that the previous statement is sharp in the sense that forbidding one more even cycle decreases the order of magnitude significantly: More generally, we show the following.

Theorem 9. *For any $k > l \geq 3$ and $m \geq 2$ such that $2k \neq ml$ we have*

$$\mathrm{ex}(n, C_{ml}, \mathscr{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

It is easy to see that forbidding even more cycles does not decrease the order of magnitude, as long as we do not forbid C_{2l} itself as shown by $(l, \lfloor n/l \rfloor)$-theta-graph and some isolated vertices, where for $l, t \geq 1$ the *(l, t)-theta-graph* with endpoints x and y is the graph obtained by joining two vertices x and y, by t internally disjoint paths of length l.

Corollary 8 determines the order of magnitude of maximum number of C_{2l}'s in a graph of girth $2l$. It is then very natural to consider the analogous question for odd cycles: What is the maximum number of C_{2k+1}'s in a graph of girth $2k+1$? Before answering this question, we state a strong form of Erdős's Girth Conjecture that is known to be true for small values of k.

A graph G on n vertices, with average degree d, is called *almost-regular* if the degree of every vertex of G is $d + O(1)$.

Conjecture 10 (Strong form of Erdős's Girth Conjecture)**.** For any positive integer k, there exists a family of almost-regular graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq \frac{n^{1+1/k}}{2}$ and G_n is $\{C_4, C_6, \dots, C_{2k}\}$-free.

Lazebnik, Ustimenko and Woldar [**16**] showed Conjecture 10 is true when $k \in \{2, 3, 5\}$ using the existence of polarities of generalized polygons. We show the following that can be seen as the 'odd cycle analogue' of Theorem 4.

Theorem 11. *Suppose $k \geq 2$ and Conjecture* 10 *is true for k. Then we have*

$$\mathrm{ex}(n, C_{2k+1}, \mathscr{C}_{2k}) = (1 + o(1)) \frac{n^{2+\frac{1}{k}}}{4k+2}.$$

To show that Theorem 11 is sharp in the same sense that Theorem 9 is (in the case of $m = 2$) for odd cycles, we prove that if we forbid one more even cycle, then the order of magnitude goes down significantly:

Theorem 12. *For any integers $k > l \geq 2$, we have*

$$\Omega(n^{1+\frac{1}{2k+1}}) = \mathrm{ex}(n, C_{2l+1}, \mathscr{C}_{2l} \cup \{C_{2k}\}) = O(n^{1+\frac{l}{l+1}}).$$

However, if the additional forbidden cycle is of odd length, we can only prove a quadratic upper bound. We conjecture that the truth is also sub-quadratic here.

Theorem 13. *For any integers $k > l \geq 2$, we have*

$$\Omega(n^{1+\frac{1}{2k+2}}) = \mathrm{ex}(n, C_{2l+1}, \mathscr{C}_{2l} \cup \{C_{2k+1}\}) = O(n^2).$$

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