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LOCALISED CODEGREE CONDITIONS FOR TIGHT HAMILTONIAN CYCLES IN 3-UNIFORM HYPERGRAPHS

P. ARAÚJO, S. PIGA AND M. SCHACHT

Abstract. We study sufficient conditions for the existence of Hamiltonian cycles in uniformly dense 3-uniform hypergraphs. Problems of this type were first considered by Lenz, Mubayi, and Mycroft for loose Hamiltonian cycles and Aigner-Horev and Levy considered it for tight Hamiltonian cycles for a fairly strong notion of uniformly dense hypergraphs. We focus on tight cycles and obtain optimal results for a weaker notion of uniformly dense hypergraphs. We show that if an n-vertex 3-uniform hypergraph $H = (V, E)$ has the property that for any set of vertices X and for any collection P of pairs of vertices, the number of hyperedges composed by a pair belonging to P and one vertex from X is at least $(1/4+o(1))|X||P|-o(|V|^3)$ and H has minimum vertex degree at least $\Omega(|V|^2)$, then H contains a tight Hamiltonian cycle. A probabilistic construction shows that the constant $1/4$ is optimal in this context.

1. Dirac's theorem and its extensions to hypergraphs

Dirac's theorem asserts that any graph $G = (V, E)$ on $|V| = n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle. This is best possible in terms of the minimum degree, since for example a graph consisting of two disjoint cliques with $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices is not even connected. We study conditions that force Hamiltonian cycles in 3-uniform hypergraphs and by a hypergraph H we always mean a 3-uniform hypergraph here.

There are several ways for defining cycles in hypergraphs and we will restrict to tight and loose cycles. A *tight* Hamiltonian cycle in a hypergraph is defined by a cyclic ordering of all vertices in which every consecutive triple of vertices forms an edge. A *loose* Hamiltonian cycle in an n-vertex hypergraph, with $n \geq 4$ even, is a cyclicly ordered collection of $n/2$ edges in such a way that two edges intersect if and only if they are consecutive and all n vertices are in at least one edge. In particular, if $n \geq 6$, then two consecutive edges must intersect in exactly one vertex. Moreover, for even n a tight cycle is the edge disjoint union of two loose cycles. There are also different notions of degree in hypergraphs. Given a

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hypergraph $H = (V, E)$ and $v \in V$, we define the *neighbourhood* and the *degree* of v by

$$N_H(v) = \{e \in E : v \in e\} \quad \text{and} \quad d_H(u) = |N(u)|\,.$$

Similarly, for two vertices u, $v \in V$, we define the *neighbourhood* and the *codegree*

$$N_H(u,v) = \{e \in E(H) : \{u,v\} \subset e\} \quad \text{and} \quad d_H(u,v) = |N(u,v)|\,.$$

Let $\delta_1(H)$ be the *minimum degree* and $\delta_2(H)$ the *minimum codegree* of H.

An extension of Dirac's theorem for hypergraphs for tight Hamiltonian cycles was proposed in [**5**] and the asymptotically optimal minimum degree and codegree conditions were obtained in [**9, 11**]. Similar results for loose Hamiltonian cycles appeared in [**3, 6**]. The lower bound constructions, which show that these results are asymptotically optimal are "very structured" in the sense, that they have some underlying vertex partition (into two parts) and there are no edges of a certain intersection type. For example, the aforementioned construction for Dirac's theorem consists of two cliques and there are no edges in between. In view of these examples, it is natural to consider extensions of those results where in the given (hyper)graph such "holes" are ruled out. This leads to the notion of uniformly dense hypergraphs.

2. Hamiltonian cycles in uniformly dense hypergraphs

For graphs, the example consisting of two cliques is excluded by the following notion of uniformly dense graphs. We say a graph $G = (V, E)$ is (ϱ, d)-*bidense* if for all subsets X and $Y \subseteq V$ we have

$$e(X,Y) = \big|\{(x,y) \in X \times Y : xy \in E\}\big| \geq d\,|X||Y| - \varrho|V|^2\,.$$

Note that this concept falls short to rule out small sets of isolated vertices, which also prevent the existence of Hamiltonian cycles. However, it is not hard to show that for every $d > 0$ every sufficiently large (ϱ, d)-bidense graph G with minimum degree $\delta(G) \geq d|V|$ contains a Hamiltionian cycle as long as ϱ is sufficiently small compared to d. This can be deduced for example from a classical result of Chvátal and Erdős [**4**] on Hamiltonian cycles.

There are several ways to define uniformly dense hypergraphs or quasirandom hypergraphs (see, e.g., [**1, 10, 12**] and the references therein for a more detailed discussion). Here we consider the following three notions.

Definition 1. For a hypergraph $H = (V, E)$ and reals d, $\varrho > 0$ we say:

- H is $(\varrho, d, \therefore)$-dense, if for all subsets X, Y, $Z \subseteq V$ we have
$$e(X,Y,Z) = \big|\{(x,y,z) \in X \times Y \times Z : \{x,y,z\} \in E\}\big| \geq d|X||Y||Z| - \varrho|V|^3.$$
- H is $(\varrho, d, \dot{\cdot\!\cdot})$-dense, if for all $X \subseteq V$, $P \subseteq V \times V$ we have
$$e(X,P) = \big|\big\{\big(x,(y,z)\big) \in X \times P : \{x,y,z\} \in E\big\}\big| \geq d|X||P| - \varrho|V|^3.$$
- H is $(\varrho, d, \wedge)$-dense, if for all P, $Q \subseteq V \times V$ we have
$$e(P,Q) = \big|\big\{\big((x,y),(y,z)\big) \in \mathcal{K}_{\wedge}(P,Q) : \{x,y,z\} \in E\big\}\big| \geq d\big|\mathcal{K}_{\wedge}(P,Q)\big| - \varrho|V|^3,$$
where $\mathcal{K}_{\wedge}(P,Q) = \big\{\big((x,y),(y',z)\big) \in P \times Q : y = y'\big\}$.

We remark that $(\varrho, d, \wedge)$-dense hypergraphs are $(\varrho, d, \dot{-})$-dense, which in turn all are $(\varrho, d, \therefore)$-dense, while there are examples that show that these inclusions are strict. For a simpler discussion we may sometimes refer to a $(\varrho, d, \star)$-dense hypergraph as a $\star$-dense hypergraph for some $\star \in \{\therefore, \dot{-}, \wedge\}$.

While the concept of $\therefore$-dense hypergraphs is a straightforward generalisation of bidense graphs, the other two concepts seem to have no direct analogue for graphs. We may think of $\dot{-}$-denseness as a *localised codegree* condition, as it ensures that a $(\varrho, d, \dot{-})$-dense hypergraph $H = (V, E)$ has the property that for any given set of vertices X, all but at most $\sqrt{\varrho}|V|^2$ pairs have at least $d|X| - \sqrt{\varrho}|V|$ neighbours inside X. Note that none of these three notions rules out the existence of isolated vertices. Consequently, for the existence of Hamiltonian cycles we may have to combine them with additional minimum degree assumptions.

This line of research was initiated by Lenz, Mubayi, and Mycroft [**7**]. Those authors showed that under the assumption $d \gg \varrho > 0$, by which we mean that the parameter ϱ is sufficiently small depending on d, any sufficiently large $(\varrho, d, \therefore)$-dense hypergraph $H = (V, E)$ with $\delta_1(H) \geq d|V|^2$ contains a loose Hamiltonian cycle. In other words, when restricting to $\therefore$-dense hypergraphs already positive density (and minimum degree of order $\Omega(|V|^2)$) suffices for the existence of loose Hamiltonian cycles.

For tight Hamiltonian cycles the situation is more involved. For this case Aigner-Horev and Levy [**2**] showed that again positive density and minimum vertex degree of order $\Omega(|V|^2)$ suffices if we consider the stronger concept of $\wedge$-dense hypergraphs. It turns out that for $\dot{-}$-density an analogous result is not possible as the following example shows.

Example 2. Consider a colouring $\varphi\colon E(K_{n-2}) \to \{\text{red}, \text{blue}\}$ chosen uniformly at random. We define a hypergraph H'_φ on the same vertex set by including $\{x, y, z\}$ as a hyperedge, if these vertices span a monochromatic triangle under φ. It can be shown that for any given $\varrho > 0$ the hypergraph H'_φ is $(\varrho, 1/4, \dot{-})$-dense with probability tending to 1 as $n \to \infty$. Moreover, any tight cycle in H'_φ can only use hyperedges with all underlying pairs having the same colour under φ. Finally, let H_φ be obtained from H'_φ by adding a vertex v_{red}, which is connected to red pairs only, and a vertex v_{blue} connected to blue pairs only. This hypergraph H_φ is still $(\varrho, 1/4, \dot{-})$-dense, it has minimum vertex degree at least $\delta_1(H_\varphi) \geq (1/4 - \varrho)n^2/2$, but no tight cycle can contain v_{red} and v_{blue} at the same time and, hence, H_φ contains no tight Hamiltonian cycle.

Our main result presented here shows that the previous random construction is essentially optimal for tight Hamiltonian cycles in $\dot{-}$-dense hypergraphs.

Theorem 3 (main result)**.** *For every $\varepsilon > 0$ there exist $\varrho > 0$ and n_0 such that every $(\varrho, 1/4 + \varepsilon, \dot{-})$-dense hypergraph H on $n \geq n_0$ vertices with $\delta_1(H) \geq \varepsilon\binom{n}{2}$ contains a tight Hamiltonian cycle.*

The proof of Theorem 3 is based on the absorption method of Rödl, Ruciński, and Szemerédi from [**11**]. Roughly speaking, that approach reduces the problem of finding a spanning Hamiltonian cycle to finding a "small" collection of vertex disjoint tight paths that cover almost all vertices of the given hypergraph H. The existence of such an almost perfect covering can be shown by fairly standard arguments already for $\therefore$-dense hypergraphs of non-vanishing density (see, e.g., [**2**] for details). Another subproblem in the absorption method concerns the abundant existence of so-called *absorbers*. For that problem we combine some ideas from [**9**] and from Polcyn and Reiher [**8**]. Again it turned out that also for this part weaker assumptions are sufficient. Here it suffices that the given hypergraph H is $\dot{-}$-dense for any density $d > 0$ and it employs the minimum vertex degree assumption. Moreover, the pieces of the almost perfect path cover and the absorbers have to be connected (to an almost spanning tight cycle). Only this part requires the full strength of the assumptions of Theorem 3 and is inspired by the lower bound construction from Example 2. Finally, the special property of the selected aborbers will be used to transform the almost spanning tight cycle into a tight Hamiltonian cycle and we omit the details here.

3. Open problems and remarks

The problems considered here concern minimum degree assumptions for uniformly dense hypergraphs that guarantee the existence of Hamiltonian cycles. The following notation will be useful for the further discussion.

Definition 4. Given $\star \in \{\therefore, \dot{-}, \boldsymbol{\wedge}\}$ and $a \in \{1, 2\}$. We say a pair of reals (d, α) is $(\star, a)$-Hamiltonian if the following assertion holds:

For every $\varepsilon > 0$ there exist $\varrho > 0$ and n_0 such that every $(\varrho, d + \varepsilon, \star)$-dense hypergraph $H = (V, E)$ with $|V| = n \geq n_0$ and $\delta_a(H) \geq (\alpha + \varepsilon)\binom{n}{3-a}$ contains a tight Hamiltonian cycle.

We remark that we can restrict our attention to tight Hamiltonian cycles, since the result of Lenz, Mubayi, and Mycroft [**7**] asserts that already $(0, 0)$ would be $(\star, a)$-Hamiltonian for loose cycles for every choice of $\star \in \{\therefore, \dot{-}, \boldsymbol{\wedge}\}$ and $a \in \{1, 2\}$. Similarly, for tight Hamiltonian cycles the work of Aigner-Horev and Levy [**2**] shows that $(0, 0)$ is $(\boldsymbol{\wedge}, a)$-Hamiltonian for $a \in \{1, 2\}$. It remains to characterise the minimal pairs (d, α) that are $(\star, a)$-Hamiltonian for the four combinations $\star \in \{\therefore, \dot{-}\}$ and $a \in \{1, 2\}$.

Example 2 shows that for (d, α) being $(\dot{-}, 1)$-Hamiltonian we must have

$$\max\{d, \alpha\} \geq \frac{1}{4}\,. \tag{1}$$

On the other hand, Theorem 3 asserts that for $d = 1/4$ already $\alpha = 0$ suffices. It would be interesting to determine the smallest value $\alpha_{\dot{-},1}$ such that $d = 0$ suffices. In view of (1) we have $\alpha_{\dot{-},1} \geq 1/4$ and the result from [**9**] bounds $\alpha_{\dot{-},1}$ by $5/9$. Since all known lower bound constructions for that result are lacking to be $\dot{-}$-dense it seems plausible that $\alpha_{\dot{-},1} < 5/9$.

Similarly, let $\alpha_{\dot{-},2}$ be the infimum over all $\alpha \geq 0$ such that $(0, \alpha)$ is $(\dot{-}, 2)$-Hamiltonian. Here it follows from [**11**] that $\alpha_{\dot{-},2} \leq 1/2$. Adding all possible triples that contain both special vertices v_{red} and v_{blue} to the hypergraph H_φ from Example 2 yields a hypergraph with minimum codegree $(1/4 - o(1))\binom{n}{2}$ and one can check that this hypergraph still fails to contain a tight Hamiltonian cycle (but the additional edges can be used to build a tight Hamiltonian path). Therefore, we have $\alpha_{\therefore,2} \geq 1/4$ and at this point we are not aware of any reason that excludes the possibility that $\alpha_{\therefore,2}$ matches this lower bound.

Problem 5. Determine $\alpha_{\therefore,1}$ and $\alpha_{\therefore,2}$.

For tight Hamilonian cycles in $\therefore$-dense hypergraphs the problem appears to be more delicate as the following unbalanced version of Example 2 shows. Instead of a uniformly chosen 2-colouring φ of $E(K_n)$ we colour edges independently red with probability p and blue with probability $1-p$. Let H_p be the resulting hypergraph, where the rest of the construction is carried out in the same way as in Example 2. By symmetry we may assume $p \geq 1/2$ and for the same reasons as for H_φ the hypergraph H_p contains no tight Hamiltonian cycle. Moreover, for every fixed $\varrho > 0$ we have with high probability that

$$\delta_1(H_p) = \big(\min\{1-p\,, p^3 + (1-p)^3\} - \varrho\big)\tbinom{n}{2}$$

and that H_p is $(\varrho, p^3 + (1-p)^3, \therefore)$-dense. For p close to 1 this shows that there is no $d < 1$ such that $(d, 0)$ is $(\therefore, 1)$-Hamiltonian. In particular, there is no straightforward analogue of Theorem 3 in this setting. Again we may add all triples containing both vertices v_{red} and v_{blue} to H_p and this adjustment yields a hypergraph $\hat{H}_p$ with the additional property

$$\delta_2(\hat{H}_p) = \big((1-p)^2 - \varrho\big)n$$

and shows that there is no $d < 1$ such that $(d, 0)$ is $(\therefore, 2)$-Hamiltonian. It would be intriguing if this construction is essentially optimal for every $p \geq 1/2$. In such an event it would imply a resolution of the following problems, where the lower bound would be obtained from $\hat{H}_p$ for $p = 2/3$ and $p = 1/2$.

Problem 6. Is it true that $(1/3, 1/3)$ is $(\therefore, 1)$-Hamiltonian?

Problem 7. Is it true that $(1/4, 1/4)$ is $(\therefore, 2)$-Hamiltonian?

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P. Araújo, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil

S. Piga, Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany

M. Schacht, Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany