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SPLITTING GROUPS WITH CUBIC CAYLEY GRAPHS OF CONNECTIVITY TWO

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Abstract. A group G splits over a subgroup C if G is either a free product with amalgamation $A \underset{C}{*} B$ or an HNN-extension $G = A \underset{C}{*} (t)$. We invoke tree-decompositions and Bass-Serre theory, and classify all infinite groups which admit cubic Cayley graphs of connectivity two in terms of splittings over a subgroup.

1. Introduction

A finitely generated group G is called *planar* if it admits a generating set S such that the Cayley graph $\mathrm{Cay}(G, S)$ is planar. In that case, S is called a *planar generating set*. For the first time, in 1896, Maschke [**12**] characterized all finite groups admitting planar Cayley graphs. Infinite planar groups attracted more attention, as some of them are related to surface and Fuchsian groups [**15**, section 4.10] which play a substantial role in complex analysis, see survey [**15**]. Hamann [**9**] uses a combinatorial method in order to show that planar groups are finitely presented. His method is based on tree-decompositions, a crucial tool of graph minor theory which we also utilize extensively in this paper.

A related topic to infinite planar Cayley graphs is the connectivity of Cayley graphs, see [**5, 7, 8**]. Studying connectivity of infinite graphs goes back to 1971 by Jung, see [**11**]. In [**5**], Droms et al. characterized planar groups with low connectivity in terms of the fundamental group of the graph of groups.

Later, Georgakopoulos [**7**] determines the presentations of all groups whose Cayley graphs are cubic with connectivity 2. His method does not assert anything regarding (and is, in a sense, independent of) splitting the group over subgroups to obtain its structure. By combining tree-decompositions and Bass-Serre theory, we give a short proof for the full characterization of groups with cubic Cayley graphs of connectivity 2 via the following theorem:

Theorem 1.1. *Let $G = \langle S \rangle$ be a group such that $\Gamma = \mathrm{Cay}(G, S)$ is a cubic graph of connectivity two. Then G is isomorphic to one of $\mathbb{Z}_n * \mathbb{Z}_2$, $D_{2n} \underset{\mathbb{Z}_2}{*} (t)$, $D_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}$, $\mathbb{Z}_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}$, $D_\infty \underset{\mathbb{Z}_2}{*} D_{2m}$.*

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As a consequence of Theorem 1.1, we obtain as a corollary the results of [**7**]. For a full version of the paper, see [**13**].

2. Preliminaries

Our terminology of groups and graphs is standard. We refer the reader to [**14**] for Bass-Serre theory and [**4**] for graph theory for any notation missing.

2.1. Graphs

Throughout this paper, Γ always denotes a connected locally finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.

A *separation* of Γ is an ordered pair (A, B) such that $\Gamma[A] \cup \Gamma[B] = \Gamma$. If $|(A, B)| = k$, we say that (A, B) is a k-separation. The set of separations of Γ can be equipped with the following partial order: $(A, B) \leq (C, D)$ if $A \subseteq C$ and $B \supseteq D$. We say that (A, B) is *nested* with (C, D) if (A, B) is comparable to either (C, D) or (D, C). A separation (A, B) *distinguishes* two ends ω_1 and ω_2 if ω_1 has a tail in $A \setminus B$ and ω_2 has a tail in $B \setminus A$ or vise versa. Moreover, it distinguishes ω_1 and ω_2 *efficiently* if there is no separation (C, D) distinguishing ω_1 and ω_2 such that $|(C, D)| < |(A, B)|$. We note that if (A, B) distinguishes two ends efficiently, then (A, B) is a tight separation. Two ends ω_1 and ω_2 are *k-distinguishable* if there is a separation of order k distinguishing ω_1 and ω_2 efficiently. Hamann et al. [**3**] proved the following theorem:

Theorem 2.1. *Let Γ be a locally finite graph with more than one end. For each $k \in \mathbb{N}$, there is a nested set $\mathcal{N}$ of tight separations of Γ distinguishing all k-distinguishable ends efficiently.* □

Let Γ be an arbitrary connected graph. A *tree-decomposition* of Γ is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(\Gamma)$, which are called *parts*, one for every node of T such that: (T1) $V(\Gamma) = \bigcup_{t \in T} V_t$, (T2) for every edge $e \in E(\Gamma)$, there exists a $t \in T$ such that both ends of e lie in V_t, (T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the (t_1, t_3)-path in T.

An *adhesion set* of $(T, \mathcal{V})$ is a set of the form $V_t \cap V_{t'}$, where $tt' \in E(T)$. We call the *torso* of a part V_t the supergraph of $G[V_t]$ obtained by adding to it all possible edges in the adhesion sets incident to V_t. It is not hard to see that each adhesion set leads to a separation of Γ and that every nested set $\mathcal{N}$ of separations gives rise to a tree-decomposition whose adhesion sets are exactly the elements of $\mathcal{N}$, see [**2**]. As an application of Theorem 2.1, one can show the following Lemma.

Lemma 2.2 ([**10**, Corollary 4.3])**.** *Let Γ be a locally finite graph with more than one end such that a group G acts on Γ. Then there exists a tree-decomposition $(T, \mathcal{V})$ with the following properties:* (i) *$(T, \mathcal{V})$ distinguishes at least two ends,* (ii) *all adhesion sets of $(T, \mathcal{V})$ are finite,* (iii) *the action of G on Γ induces an action on $\Gamma[\mathcal{V}]$ and a transitive action on the set of separations corresponding to the adhesion sets.*

Notice that the transitive action on the set of separations in Lemma 2.2 (iii) implies at most two orbits for $\Gamma(\mathcal{V})$ under the action of G. Moreover, we can translate the action of item (iii) to an action of G on T in the natural way (and G will clearly act transitively on $E(T)$): $gt = t' \Leftrightarrow gV_t = V_{t'}$.

In this paper, we are studying groups admitting cubic Cayley graphs of connectivity two. The next Lemma implies that such a graph has at least two ends.

Lemma 2.3 ([**1**, Lemma 2.4])**.** *Let* Γ *be a connected vertex-transitive* d*-regular graph. Assume* Γ *has one end. Then the connectivity of* Γ *is* $\geq 3(d+1)/4$.

2.2. Groups

Let G be a group acting on a set X. Then the *setwise stabilizer* of a subset Y of X is the set of all elements $g \in G$ stabilizing Y setwise, i.e $\mathsf{St}_G(Y) := \{g \in G \mid gy \in Y, \forall y \in Y\}$. Let G be a group acting on a graph Γ. Then this action induces an action on $E(\Gamma)$. We say that G acts without *inversion* on Γ if $g(uv) \neq vu$ for all $uv \in E(\Gamma)$ and $g \in G$. In the case that $g(uv) = vu$, we say that g *inverts* u, v. Notice that when G acts transitively with inversion on the set $E(T)$ of edges of a tree T without leaves, it must also act transitively on the set $V(T)$ of its vertices.

Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ be two groups. Suppose that a subgroup H_1 of G_1 is isomorphic to a subgroup H_2 of G_2, say an isomorphic map $\phi\colon H_1 \to H_2$. The *free-product with amalgamation* of G_1 and G_2 over H_1 is $G_1 \underset{H_1}{*} G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup h\phi(h)^{-1}, \forall h \in H_1 \rangle$.

If H_1 and $\phi(H_1)$ are isomorphic subgroups of G_1, then the HNN-*extension* of G_1 over H_1 with respect to ϕ is $G_1 \underset{H_1}{*} (t) = \langle S_1, t \mid R_1 \cup tht^{-1}\phi(h)^{-1}, \forall h \in H_1 \rangle$.

The crux of Bass-Serre theory is captured in the next Lemma which determines the structure of groups acting on trees.

Lemma 2.4 ([**14**])**.** *Let* G *act without inversion on a tree that has no vertices of degree one and let* G *act transitively on the set of (undirected) edges. If* G *acts transitively on the tree, then* G *is an* HNN*-extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are two orbits on the vertices of the tree, then* G *is the free product of the stabilizers of two adjacent vertices with amalgamation over the pointwise stabilizer of an edge.*

Finally, $\mathbb{Z}_n$ denotes the *cyclic group of order* n. A finite *dihedral* group is defined by the presentation $\langle a, b \mid b^2 = a^n = (ba)^2 \rangle$ and denoted by D_{2n}. Moreover, the infinite dihedral group D_∞ is defined by $\langle a, b \mid b^2 = (ba)^2 \rangle$.

3. General structure of the tree-decomposition

For the rest of the paper, we assume that $G = \langle S \rangle$ is an infinite finitely generated group such that $\Gamma = \mathrm{Cay}(G, S)$ is cubic with connectivity two. Let $\mathcal{N}$ be a nested set of separations of order two in such a way that $\mathcal{N}$ gives a tree-decomposition as in Lemma 2.2. Then we notice that every 2-separation of Γ such that $A \cap B$ is a proper subset of A and B distinguishes at least two ends, see [**6**, Lemma 3.4]. For an arbitrary element $(A, B) \in \mathcal{N}$, there are three cases as in Fig. 1.

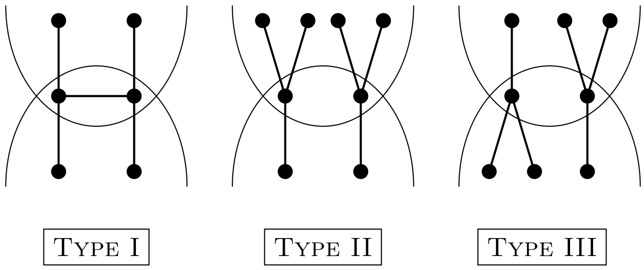

Figure 1. The three types of splitting 2-separations in cubic Cayley graphs of connectivity 2.

First, we dismiss the case of Type III separations by easily showing that we can always choose Type II instead for the nested set of separations and the respective tree-decomposition obtained by Lemma 2.1 and Lemma 2.2.

Lemma 3.1. *Assume that* Γ *contains a* Type III *separation distinguishing efficiently at least two ends. Then it also contains a* Type II *separation distinguishing efficiently the same ends.*

In what follows, $(T, \mathcal{V})$ will always be as in Lemma 2.2, either of Type I or Type II if not specified. For a node $t \in V(T)$, we define $n(t) := \Gamma\left[\bigcup_{t\in N_T[t]} V_t\right]$.

Let H be an arbitrary graph with a set $U \subseteq V(H)$ and a subgraph H' of H. The set U is called *connected in* H' if for every pair of vertices $u, u' \in U$ there is a (u, u')-path in H'.

Lemma 3.2. *Let* t *be an arbitrary vertex of* T*. Then for every* $t' \in N_T(t)$*,* (i) *the adhesion set* $V_t \cap V_{t'}$ *is connected in at least one of* $V_t, V_{t'}$*,* (ii) V_t *is connected in* $n(t)$*.*

The next crucial lemma implies that all adhesion sets in $\mathcal{N}$ are disjoint.

Lemma 3.3. *Let* t *be a node of* T*. Then for every* $t_1, t_2 \in N_T(t)$*, we have* $V_{t_1} \cap V_{t_2} = \emptyset$*.*

Corollary 3.4. *Every vertex* u *of* Γ *is contained in exactly two parts* $t, t' \in V(T)$*. In addition,* $N_\Gamma(u) \subseteq V_t \cup V_{t'}$ *and every part is the disjoint union of its adhesion sets.*

Moreover, let $\{x, y\}$ be an adhesion set. Observe that $xy^{-1}\{x, y\}$ is again an adhesion set containing x, so $xy^{-1}\{x, y\} = \{x, y\}$ with $xy^{-1}x = y$. We obtain:

Lemma 3.5. *For every adhesion set* $\{x, y\}$*, we have* $(xy^{-1})^2 = 1$*.*

Corollary 3.6. *Let* $tt' \in E(T)$*. Then* $\mathsf{St}_G(V_t \cap V_{t'}) \cong \mathbb{Z}_2$*.*

4. The case of a tree-decomposition of Type II and three generators

In this extended abstract, we briefly discuss only the case when G is generated by three involutions and the tree-decomposition of the respective Cayley graph is of Type II. For a full discussion of all cases, we refer the reader to the full version [**13**]. Let $G = \langle a, b, c\rangle$, where a, b and c are involutions. Then – up to rearranging a, b, c – we have two cases for the separations in $\mathcal{N}$ as in Fig 2. We can prove that we actually have one.

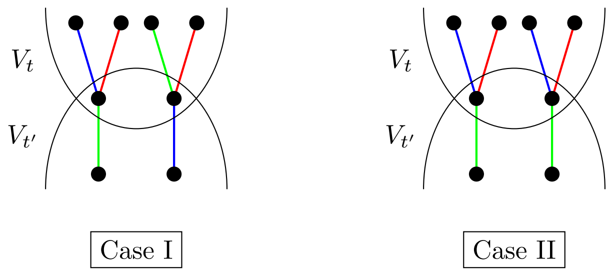

Figure 2. Type II cases with three generators

Lemma 4.1. *The adhesion sets of* $(T, \mathcal{V})$ *satisfy Case II.*

Since the torso of every part of $(T, \mathcal{V})$ is a connected graph, we deduce that the tree-decomposition has two orbits of parts: parts in O_1 contain only b- and c-edges and parts in O_2 induce perfect a-matchings. Clearly, G then acts on $(T, \mathcal{V})$ without inversion.

Lemma 4.2. *Every part in* O_1 *induces an alternating* (b, c)*-cycle of length a multiple of* 4 *or an alternating double* (b, c)*-ray.*

By the 2-connectivity of Γ, the connected, 2-regular torso of a part $V_s \in O_2$ must be a finite cycle. Depending on which of the cases of Lemma 4.2 we have, we can label its edges with $(bc)^n$ or $(bc)^n b$ (corresponding to the virtual edges of the torso) and a in an alternating fashion. Therefore, there is an $m \geq 2$ such that $(a(bc)^n)^m = 1$ or $(a(bc)^n b)^m = 1$. It remains to infer the structure of G in each case.

Suppose that every part in O_1 is an alternating (b, c)-cycle of length $4n$ and $(a(bc)^n)^m = 1$. We can see that $\mathsf{St}_G(V_{t_1}) = V_{t_1} \cong D_{4n}$ and $\mathsf{St}_G(V_{t_2}) = V_{t_2} = \langle a(bc)^n, a \mid (a(bc)^n)^m, a^2, (a(a(bc)^n))^2\rangle \cong D_{2m}$ to infer by Lemma 2.4 that $G \cong D_{4n} \underset{\mathbb{Z}_2}{*} D_{2m}$.

Assume that every part in O_1 is an alternating double (b, c)-ray and that $(a(bc)^n b)^m = 1$. Similarly, $\mathsf{St}_G(V_{t_1}) = V_{t_1} = \langle bc, b \mid b^2, (b(bc))^2\rangle \cong D_\infty$ and $\mathsf{St}_G(V_{t_2}) = V_{t_2} = \langle a(bc)^n b, a \mid (a(bc)^n b)^m, a^2, (a(a(bc)^n b))^2\rangle \cong D_{2m}$ to deduce by Lemma 2.4 that $G \cong D_{4n} \underset{\mathbb{Z}_2}{*} D_{2m}$.

Theorem 4.3. *If $(T, \mathcal{V})$ is of* Type II *with three generators, then* $G \cong D_{4n} \underset{\mathbb{Z}_2}{*} D_{2m}$ *or* $G \cong D_\infty \underset{\mathbb{Z}_2}{*} D_{2m}$.

Corollary 4.4 ([**7**, Theorem 1.1])**.** *If $(T, \mathcal{V})$ is of* Type II *with three generators, then either $G = \langle a, b, c \mid a^2, b^2, c^2, (bc)^{2n}, (a(bc)^n)^m \rangle$ and Γ is planar if and only if $n = 1$, or $G = \langle a, b, c \mid a^2, b^2, c^2, (a(bc)^n b)^m \rangle$ and Γ is planar if and only if $n = 1$.* □

5. Open questions

Some further open questions can naturally be raised. In light of Lemma 2.3, we can ask the following.

Problem 1. Characterize all groups admitting 4-regular Cayley graphs of connectivity at most three in terms of splitting over subgroups.

A graph is called quasi-transitive if it has a finite number of orbits of vertices under the action of its automorphism group. Looking back at Theorem 1.1, we see that cubic Cayley graphs of connectivity two can be expressed as a tree decomposition whose torsos induce two cycles or the double ray and a cycle. The main combinatorial tools from our proof follow through to show that this is in general the case for every cubic transitive graph of connectivity two. We can go a step futher and ask the following question:

Problem 2. Characterize all cubic quasi-transitive graphs of connectivity two in terms of "canonical" tree decompositions with the property that the automorphism group of the graph acts transitively on the set of the adhesion sets.

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