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DOUBLY BIASED WALKER-BREAKER GAMES

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ABSTRACT. We study doubly biased Walker-Breaker games, played on the edge set of a complete graph on n vertices, K_n. Walker-Breaker game is a variant of Maker-Breaker game, where Walker, playing the role of Maker, must choose her edges according to a walk, while Breaker has no restrictions on choosing his edges. Here we show that for $b \leq \frac{n}{10 \ln n}$, playing a $(2 : b)$ game on $E(K_n)$, Walker can create a graph containing a spanning tree. Also, we determine a constant $c > 0$ such that Walker has a strategy to make a Hamilton cycle of K_n in the $(2 : \frac{cn}{\ln n})$ game.

1. INTRODUCTION

We study a variant of the well known Maker-Breaker positional games on graphs. Let X be a finite set of elements, that we call the *board* of the game, let $\mathcal{F} \subseteq 2^X$ be the family of *winning sets*, and let a and b be two positive integers. In the $(a : b)$ Maker-Breaker game on X, two players, Maker and Breaker, alternate in claiming a (respectively b) unclaimed elements of the board, with Breaker going first. The game ends when all elements are claimed. Maker wins the game if by the end of the game she has claimed all the elements of one winning set, and Breaker wins otherwise. When $a = b = 1$, the games are called *unbiased*, and otherwise are called *biased* games.

The standard approach is to look at the Maker-Breaker games played on the edge set of the complete graph on n vertices, K_n, i.e. $X = E(K_n)$. The winning sets are some graph theoretic properties, such as spanning trees, perfect matchings, Hamilton cycles, cliques, etc. In the *Connectivity game* the winning sets are the edge sets of the spanning trees of K_n and in the *Hamiltonicity game* the winning sets are the edge sets of all Hamilton cycles in K_n. There has been a lot of research in this aspect in recent years, and lots of examples of Maker-Breaker games can be found in the book of Beck[**1**] and in the recent monograph of Hefetz, Krivelevich, Stojaković and Szabó [**6**]. Regarding unbiased games, Lehman showed in [**9**] that for $n \geq 4$, Maker easily wins the unbiased Connectivity game on K_n. Also, it is shown in [**7**] that Maker is able to create a Hamilton cycle, within $n+1$ rounds. In order to give some advantage to Breaker, the biased $(1 : b)$ games were introduced by Chvátal and Erdős in [**2**]. Other way to provide greater chances for Breaker to win is to restrict Maker's selections of edges in graph. In Walker-Breaker game,

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introduced by Espig, Frieze, Krivelevich and Pegden [**4**], Walker is not allowed to choose any free edge in graph. Instead, she has to choose her edges according to a walk, that is, in each move she needs to choose an edge incident with some vertex v in which she is currently positioned, that is not already claimed by Breaker (but could have been chosen by Walker earlier). Breaker, on the other hand, can claim any edge not already claimed.

Not so much is known about Walker-Breaker games. It is easy to see that Walker cannot make a spanning tree, nor any spanning structure in the unbiased Walker-Breaker game on K_n, and thus in the biased game as well. Clemens and Tran in [**3**] considered how large a cycle can Walker make in both unbiased and biased Walker-Breaker game on K_n, and proved that Walker can create a cycle of length $n-2$ in the unbiased Walker-Breaker game, while in the biased $(1 : b)$ Walker-Breaker game, Walker can create a cycle of length $n - O(b)$ for $b \leq \frac{n}{\ln^2 n}$. They also posted the following two questions.

Question 1.1 ([**3**, Problem 6.4])**.** What is the largest bias b for which Walker has a strategy to create a spanning tree of K_n in the $(2 : b)$ Walker-Breaker game on K_n?

Question 1.2 ([**3**, Problem 6.5])**.** Is there a constant $c > 0$ such that Walker has a strategy to occupy a Hamilton cycle of K_n in the $(2 : \frac{cn}{\ln n})$ Walker-Breaker game on K_n?

Our research is focused on the biased $(2 : b)$ Walker-Breaker Connectivity and Hamiltonicity games and here, we address both of the mentioned questions.

2. Preliminaries

Our notation is standard and follows that of [**10**]. Specifically, we use the following.

For given graph G by $V(G)$ and $E(G)$ we denote its vertex set and edge set, respectively. The order of graph G is denoted by $v(G)$, and the size of the graph by $e(G)$. Let $d_G(v)$ denote the degree of vertex v in G and $d_G(v, B)$ the degree of vertex v in G towards vertices from B. Assume that the Walker-Breaker game, played on the edge set of graph K_n, is in progress. At any given moment during this game, we denote the graph spanned by Walker's edges by W and the graph spanned by Breaker edges by B. At any point during the game, the edges of $K_n \setminus (W \cup B)$ are called *free*.

Let n be a positive integer and let $0 < p < 1$. The Erdős-Rényi model $G(n, p)$ is a random subgraph G of K_n, constructed by retaining each edge of K_n in G independently at random with probability p. We say that graph $G(n, p)$ possesses a graph property $\mathcal{P}$ *asymptotically almost surely*, or a.a.s., for brevity, if the probability that $G(n, p)$ possesses $\mathcal{P}$ tends to 1 as n goes to infinity.

In order to answer the Question 1.2, we need some statements related to local resilience and random graphs.

Definition 2.1 ([**3**]). For $n \in \mathbb{N}$, let $\mathcal{P} = \mathcal{P}(n)$ be some graph property that is monotone increasing, and let $0 \le \varepsilon, p = p(n) \le 1$. Then $\mathcal{P}$ is said to be (p, ε)-resilient if a random graph $G \sim \mathbb{G}(n,p)$ a.a.s. has the following property: For every $R \subseteq G$ with $d_R(v) \le \varepsilon d_G(v)$ for every $v \in V(G)$ it holds that $G \setminus R \in \mathcal{P}$.

Next theorem provides a good bound on the local resilience of a random graph with respect to the Hamiltonicity.

Theorem 2.2 ([**8**]). *For every positive $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for $p \ge \frac{C \ln n}{n}$, a graph $G \sim \mathbb{G}(n,p)$ is a.a.s. such that the following holds. Suppose that H is a subgraph of G for which $G' = G - H$ has minimum degree at least $(1/2 + \varepsilon)np$, then G' is Hamiltonian.*

3. Results

First, we look at the Question 1.1. Our first result shows that Walker can make a spanning tree in Walker-Breaker Connectivity game on K_n even if Breaker's bias is $b = \frac{n}{10 \ln n}$.

Theorem 3.1. *For every large enough n and $b \le \frac{n}{10 \ln n}$, Walker has a strategy to win in the biased $(2 : b)$ Walker-Breaker Connectivity game played on K_n.*

Next, we look at the $(2 : b)$ Walker-Breaker Hamiltonicity game on K_n and answer the Question 1.2 affirmatively.

Theorem 3.2. *There exists a constant $c > 0$ for which for every large enough n and $b \le \frac{cn}{\ln n}$, Walker has a winning strategy in the $(2 : b)$ Hamiltonicity game played on $E(K_n)$.*

4. Concluding remarks

When Walker's bias is 1, she cannot hope to make a spanning structure on K_n even in the unbiased Walker-Breaker game. Our results show that increasing Walker's bias by just one, she is able to make a spanning structure even when playing against Breaker whose bias is of the order $\frac{n}{\ln n}$.

Since now we know that Walker can win both Connectivity and Hamiltonicity game, one research direction for further investigation would be to find a strategy which allows her to win in the k-vertex connectivity game, for $k > 1$.

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