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# TWO VALUES OF THE CHROMATIC NUMBER OF A SPARSE RANDOM GRAPH

S. KARGALTSEV, D. SHABANOV AND T. SHAIKHEEVA

Abstract. The famous results of Łuczak (1991) and Alon – Krivelevich (1997) state that the chromatic number $\chi(G(n,p))$ of the binomial random graph $G(n,p)$ is concentrated in two consecutive values with probability tending to 1 provided $p = p(n) \le n^{-1/2-\varepsilon}$. Unfortunately, their proofs do not give the explicit values of $\chi(G(n,p))$ as functions of $n$ and $p$. Achlioptas and Naor (2005) found these values in the sparse case when $np$ is fixed. Coja-Oghlan, Panagiotou and Steger (2008) showed that the chromatic number of $G(n,p)$ is concentrated in three explicit consecutive values provided $p = p(n) \le n^{-3/4-\delta}$, they also established a 2-point concentration for the "half" of the values of the parameter $p$ under these conditions. In the current paper we improve the discussed result and show that the concentration of the chromatic number in two explicit consecutive values holds "almost everywhere" provided $p = p(n) \le n^{-3/4-\delta}$ and $np \to +\infty$. Namely, if $r_p = \min\{r : (n-1)p < 2r\ln r\}$ then we prove that for

$$(n-1)p \in \left(2(r_p-1)\ln(r_p-1), 2r_p\ln r_p - \ln r_p - 2 - r_p^{-1/6}\right),$$

it holds that

$$\mathsf{Pr}\left(\chi(G(n,p)) \in \{r_p, r_p+1\}\right) \to 1 \text{ as } n \to +\infty.$$

## 1. Introduction

The paper deals with the well–known problem concerning the chromatic number of a random graph. Let $G(n,p)$ denote the binomial model of a random graph, in which every edge of the complete graph on $n$ vertices is included into $G(n,p)$ independently with probability $p$.

The problem of estimating the chromatic number of $G(n,p)$ has a huge background, it was intensively studied for the decades. The first sharp asymptotics for $\chi(G(n,p))$ was established by Bollobás [**4**] in the dense case. For a fixed $p \in (0,1)$, he showed that

$$\chi(G(n,p)) \cdot \left(\frac{n}{2\log_{(1-p)^{-1}} n}\right)^{-1} \xrightarrow{\mathsf{Pr}} 1 \qquad \text{as } n \to +\infty. \tag{1}$$

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The argument of the above result also works for a slowly enough decreasing function $p = p(n)$. The remaining regimes were investigated by Łuczak [**14**], who showed that if $p = p(n) = o(1)$ but $np \to +\infty$ with growth of $n$ then

$$\chi(G(n,p)) \cdot \left(\frac{np}{2\ln(np)}\right)^{-1} \xrightarrow{\mathsf{Pr}} 1 \qquad \text{as } n \to +\infty.$$

Recent refinements of (1) in the dense case have been obtained by Heckel [**9**]. Advances concerning the chromatic number of dense random subgraphs of Knezer graphs and hypergraphs can be found, e.g., in [**10**], [**13**].

Another remarkable result of Łuczak [**15**] states that for $p \le n^{-5/6}$, there is a concentration of $\chi(G(n,p))$ in two consecutive values with probability tending to 1, i.e., there exists a function $h = h(n)$ such that

$$\mathsf{Pr}\left(\chi(G(n,p)) \in \{h, h+1\}\right) \to 1 \qquad \text{as } n \to \infty. \tag{2}$$

Alon and Krivelevich [**3**] showed that the same situation holds up to $p = p(n) \le n^{-1/2-\varepsilon}$ where $\varepsilon > 0$ is an arbitrary positive constant. However, the proofs of these results do no give any reasonable information about the exact value of the function $h$ in (2). The first advancement in this direction was made by Achlioptas and Naor [**1**] for the sparse case when $np = c > 0$ is a fixed number. Their theorem can be formulated as follows: suppose that $c > 0$ is fixed and let $r_c$ denote the smallest natural number $r$ such that $c < 2r\ln r$, then

$$\mathsf{Pr}\left(\chi(G(n,c/n)) \in \{r_c, r_c+1\}\right) \to 1 \qquad \text{as } n \to +\infty.$$

This result completely solves the question concerning two-point concentration, however since the $r$-colorability property has a sharp threshold (see [**2**]), $r \ge 3$, we should expect one-point concentration instead. It is easy to verify that for $c > 2r\ln r - \ln r$, the random graph $G(n,c/n)$ is not $r$-colorable with probability tending to 1. This observation together with the result of Achlioptas and Naor yields that for $c \in (2r\ln r - \ln r, 2r\ln r)$, the chromatic number of $G(n,p)$ is exactly equal to $r+1$ with probability tending to 1, but for $c \in (2(r-1)\ln(r-1), 2r\ln r - \ln r)$, it is equal to $r$ or $r+1$. The best current estimates of the $r$-colorability thresholds were obtained by Coja-Oghlan and Vilenchik ([**6**], lower bound) and by Coja-Oghlan ([**7**], upper bound). Their results state that

- if $c < 2\ln r - \ln r - 2\ln 2 - o_r(1)$ then $\mathsf{Pr}\left(\chi(G(n,c/n)) \le r\right) \to 1$ as $n \to +\infty$;
- if $c > 2\ln r - \ln r - 1 + o_r(1)$ then $\mathsf{Pr}\left(\chi(G(n,c/n)) > r\right) \to 1$ as $n \to +\infty$.

So, when the expected number of edges is linear, $p = c/n$, only in the short intervals of the type $c \in (2\ln r - \ln r - 2\ln 2 - o_r(1), 2\ln r - \ln r - 1 + o_r(1))$ we do not know the exact limit value of the chromatic number of $G(n,c/n)$.

The first attempt to find the values of $\chi(G(n,p))$ for growing $np$ was made in the paper [**8**] by Coja-Oghlan, Panagiotou and Steger. For not too large $p = p(n)$, they were able to establish the concentration of the chromatic number in three consecutive explicit values. The exact formulation is the following: suppose

$0 < \delta \le 1/4$ is fixed and $p \le n^{-3/4-\delta}$, denote $r_p = r_p(n) = \min\{r : p(n-1) < 2r \ln r\}$, then

$$\mathsf{Pr}\left(\chi(G(n,p)) \in \{r_p, r_p+1, r_p+2\}\right) \to 1 \text{ as } n \to +\infty.$$

Moreover, if $p(n-1) > 2r_p \ln r_p - \ln r_p + \varepsilon$ for a fixed $\varepsilon > 0$ then

$$\mathsf{Pr}\left(\chi(G(n,p)) \in \{r_p+1, r_p+2\}\right) \to 1 \text{ as } n \to +\infty.$$

Thus, in the interval $p(n-1) \in (2(r_p-1)\ln(r_p-1), 2r_p \ln r_p)$ the function $h(n)$ from (2) is exactly known roughly for half of the values.

## 2. New result

Our main result refines the theorem of Coja-Oghlan, Panagiotou and Steger and provides the exact values of two-point concentration for the almost all remaining situations.

**Theorem 2.1.** *Suppose that* $0 < \delta \le 1/4$ *is fixed and* $p \le n^{-3/4-\delta}$. *Let us denote* $r_p = r_p(n) = \min\{r : p(n-1) < 2r \ln r\}$. *If*

$$p(n-1) < 2r_p \ln r_p - \ln r_p - 2 - r_p^{-1/6}$$

*then*

$$\mathsf{Pr}\left(\chi(G(n,p)) \in \{r_p, r_p+1\}\right) \to 1 \textit{ as } n \to +\infty.$$

Together with the previous results Theorem 2.1 states that we do not have concentration in two explicit consecutive numbers only in the situation when $p(n-1)$ lies in the interval of a bounded length, namely

$$p(n-1) \in [2r_p \ln r_p - \ln r_p - 2 - r_p^{-1/6}, 2r_p \ln r_p - \ln r_p + \varepsilon].$$

This is quite similar to the case when $pn$ is fixed, but everytime we need one more color.

## 3. Ideas of the proof

The proof of Theorem 2.1 follows the general scheme from [**8**] and starting from some moment we can just repeat the arguments.

### 3.1. Second moment method

The key ingredient of the argument from [**8**] is the following technical theorem from the paper of Achlioptas and Naor [**1**]. Let $\mathcal{D}_r$ denote the set of $r \times r$ matrices $M = (m_{ij}, i, j = 1, \ldots, r)$ with nonnegative elements satisfying the following conditions:

$$\sum_{i=1}^{r} m_{ij} = \frac{1}{r}, \text{ for any } j = 1, \ldots, r; \quad \sum_{j=1}^{r} m_{ij} = \frac{1}{r}, \text{ for any } i = 1, \ldots, r.$$

For any $M \in \mathcal{D}_r$, denote

$$\mathcal{H}(M) = -\sum_{i,j=1}^{r} m_{ij} \ln m_{ij}; \quad \mathcal{E}(M) = \ln\left(1 - \frac{2}{r} + \sum_{i,j=1}^{r} m_{ij}^2\right).$$

Denote for $d > 0$, $\mathcal{G}_d(M) = \mathcal{H}(M) + d \cdot \mathcal{E}(M)$. The result of Achlioptas and Naor states that for $d < 2(r-1)\ln(r-1)$, the value $\mathcal{G}_d(M)$ reaches its maximum value at the matrix $J_r$ which has all entries equal to $1/r^2$. We improve this assertion as follows.

**Lemma 3.1.** *There exists an absolute constant $r_0$ such that for any $r > r_0$, $d < 2r\ln r - \ln r - 2 - r^{-1/6}$ and any $M \in \mathcal{D}_r$, we have $\mathcal{G}_d(M) \le \mathcal{G}_d(J_r)$.*

The proof of Lemma 3.1 follows the analysis from the papers [**11, 12, 16, 17**] concerning colorings of random $k$-uniform hypergraphs. E.g., in [**11**] the second moment method was used to obtain very tight estimates for the panchromatic 3-colorability threshold in a random $k$-uniform hypergraph (recent advances on panchromatic colorings of hypergraphs can be found in [**5**]). However, the proofs in [**11, 12, 16, 17**] hold only for $k \ge 4$, so we were not able to apply them directly to the case of graphs and had to derive some new ideas.

As a corollary of Lemma 3.1 we obtain the following result for the case of fixed $np$.

**Corollary 3.2.** *Suppose that $np = c > 0$ is fixed. There exists an absolute constant $r_0$ such that for any $r > r_0$, $c < 2r\ln r - \ln r - 2 - r^{-1/6}$, we have*

$$\mathsf{Pr}\left(\chi(G(n, c/n)) \le r\right) \to 1 \qquad \text{as } n \to +\infty.$$

Note that this bound for $r$-colorability threshold is slightly weaker than the result obtained in [**6**], but it also has a constant gap with the known upper bound.

Lemma 3.1 helps to estimate the second moment of the number of proper balanced $r$-colorings (i.e. colorings with almost equal sizes of color classes) of a random graph in the uniform model $G(n,m)$ in which $m$ edges are chosen randomly without replacement and $m = \lfloor p\binom{n}{2}\rfloor$. Together with Proposition 3.3 and Lemma 3.4 from [**8**] it implies that under the condition of Theorem 2.1 it holds that

$$\mathsf{Pr}\left(\chi(G(n,p)) \le r_p\right) \ge \mathrm{e}^{-6(np)^2} n^{-r_p^2}. \tag{3}$$

### 3.2. Completion of the proof

From this moment one can just repeat the argument from [**8**], so we will not go into any details and will briefly describe the approach.

The inequality (3) shows that $r_p$ colors are not enough to color $G(n,p)$ properly. However, Lemma 4.2 from [**8**] states that with probability tending to 1 we can color properly almost the whole graph except the vertex subset $U_0$ of the size at most $n^{3/2}p\ln n$. For small $p \le n^{-1+1/20}$, we can show that there exists a larger subset $U \supset U_0$ such that

- the subgraph induced on $U$ is 3-colorable;
- its neighborhood $N(U)$ in $G(n,p)$ is an independent set.

Thus, we color $U$ with colors $\{1,2,3\}$, $G(n,p)\setminus(U\cup N(U))$ with colors $\{1,\dots,r_p\}$ and $N(U)$ with color $r_p+1$.

For large $p\in[n^{-1+1/20},n^{-3/4-\delta}]$, one can use the approach of Alon and Krivelevich [**3**] to find a subset $U\supset U_0$ such that $G(n,p)\setminus U$ is still $r_p$-colorable and every vertex outside $U$ has small number of neighbors in $U$. After that we can modify a proper $r$-coloring of $G(n,p)\setminus U$ with additional color $r_p+1$ to get a small enough number of restrictions for coloring of $U$ which can be colored with the obtained list coloring by the help of the Local Lemma.

### 3.3. Ideas of the proof of Lemma 3.1

In this section we comment on the proof of Lemma 1. To show that $\mathcal{G}_d(M)\le\mathcal{G}_d(J_r)$ we consider this difference "by rows":

$$\begin{aligned}\mathcal{G}_d(J_r)-\mathcal{G}_d(M)&=\sum_{i,j=1}^{r}m_{ij}\ln m_{ij}-d\ln\left(1-\frac{2}{r}+\sum_{i,j=1}^{r}m_{ij}^2\right)\\&\quad-2\ln r+d\ln(1-1/r)^2=\\&=\sum_{i,j=1}^{r}m_{ij}\ln(r^2m_{ij})-d\ln\left(1+\frac{\sum_{i,j=1}^{r}m_{ij}^2-1/r^2}{(1-1/r)^2}\right).\end{aligned}$$

Now denote

$$g_i(M)=\sum_{j=1}^{r}m_{ij}\ln(r^2m_{ij})-d\left(\frac{\sum_{j=1}^{r}m_{ij}^2-1/r^4}{(1-1/r)^2}\right).$$

A row $m_i=(m_{ij},j=1,\dots,r)$ is said to be

- *good* if every $m_{ij}\le\frac{1}{r}-\frac{2}{r\ln r}$;
- *normal* if there exists $j$ such that $m_{ij}\in[\frac{1}{r}-\frac{2}{r\ln r};\frac{1}{r}-r^{-7/4}]$;
- *bad* if there exists $j$ such that $m_{ij}\in[\frac{1}{r}-r^{-7/4};1/r]$.

We prove that for any good row $m_i$,

$$g_i(M)\ge\frac{r^2}{4}\sum_{j:m_{ij}<1/r^2}\left(m_{ij}-\frac{1}{r^2}\right)^2+\frac{r}{2}\sum_{j:m_{ij}>1/r^2}\left(m_{ij}-\frac{1}{r^2}\right)^2;$$

and for any normal row

$$g_i(M)\ge\frac{1}{8}r^{-7/4}\ln r.$$

Finally, we consider the bad rows simultaneously. Let $I\subset\{1,\dots,r\}$ denote the set of indexes corresponding to the bad rows. Then we show that

$$\begin{aligned}Z_I&=\sum_{i\in I}\sum_{j=1}^{r}m_{ij}\ln(r^2m_{ij})-d\ln\left(1+\frac{\sum_{i\in I}\sum_{j=1}^{r}m_{ij}^2-1/r^2}{(1-1/r)^2}\right)\\&\ge\frac{|I|\ln r}{2r^2}\left(\frac{|I|}{r}-1\right)+\frac{|I|}{r^{2+1/6}}-O\left(\frac{\ln r}{r^{3/2}}\right).\end{aligned}$$

Note that
$$\mathcal{G}_d(J_r) - \mathcal{G}_d(M) \geq \sum_{i \notin I} g_i(M) + Z_I.$$
In the final part of the proof we consider different cases for the number of bad rows $|I|$ and show that the above estimate is always positive.

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S. Kargaltsev, Moscow Institute of Physics and Technology, Department of Data Analysis, Dolgoprudny, Moscow Region, Russia,
*e-mail*: `stepan.kargaltsev@phystech.edu`

D. Shabanov, Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia,
*e-mail*: `dmitry.shabanov@phystech.edu`

T. Shaikheeva, Department of Probability Theory, Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow, Russia,
*e-mail*: `talia.shaikheeva@yandex.ru`