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MINIMUM DEGREE CONDITIONS FOR POWERS OF CYCLES AND PATHS

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Abstract. The study of conditions on vertex degrees in a host graph G for the appearance of a target graph H is a major theme in extremal graph theory. The k^{th} power of a graph F is obtained from F by joining any two vertices at distance at most k. We study minimum degree conditions under which a graph G contains the k^{th} power of cycles and paths of arbitrary specified lengths. We determine precise thresholds, assuming that the order of G is large. This extends a result of Allen, Böttcher and Hladký concerning the containment of squared paths and squared cycles of arbitrary specified lengths and settles a conjecture of theirs in the affirmative.

1. Introduction

The study of conditions on vertex degrees in a host graph G for the appearance of a target graph H is a major theme in extremal graph theory. One of the best-known results in this area is the following theorem of Dirac about the existence of a Hamiltonian cycle.

Theorem 1.1 (Dirac [**2**]). *Every graph on $n \geq 3$ vertices with minimum degree at least $\frac{n}{2}$ has a Hamiltonian cycle.*

The k^{th} power of a graph G, denoted by G^k, is obtained from G by joining any two vertices at distance at most k. In 1962, Pósa conjectured an analogue of Dirac's theorem for the containment of the square of a Hamiltonian cycle. This was extended in 1974 by Seymour to general powers of a Hamiltonian cycle.

Conjecture 1 (Pósa–Seymour). Let $k, n \in \mathbb{N}$. A graph on n vertices with minimum degree at least $\frac{kn}{k+1}$ contains the k^{th} power of a Hamiltonian cycle.

Fan and Kierstead made significant progress, proving an approximate version of this conjecture for squared paths and squared cycles in sufficiently large graphs [**3**] and determining the best-possible minimum degree condition for a Hamiltonian squared path [**4**]. Komlós, Sárközy and Szemerédi confirmed the truth of the Pósa–Seymour Conjecture for sufficiently large graphs.

Theorem 1.2 (Komlós–Sárközy–Szemerédi [**6**]). *For every positive integer k, there exists a positive integer $n_0 = n_0(k)$ such that for all positive integers $n > n_0$,*

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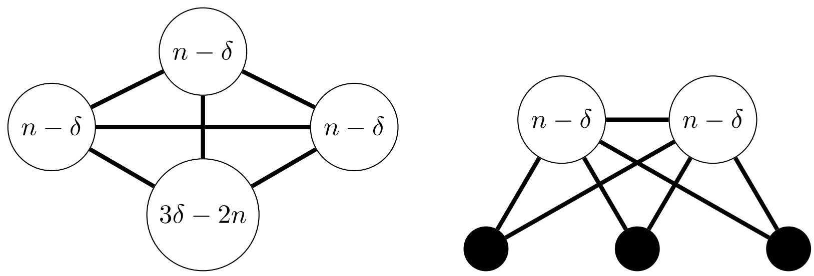
any graph G on n vertices with minimum degree at least $\frac{kn}{k+1}$ contains the k^{th} power of a Hamiltonian cycle.

In fact, the proof asserts a stronger result, guaranteeing the k^{th} power of cycles of all lengths between $k+1$ and n which are divisible by $k+1$, in addition to the k^{th} power of a Hamiltonian cycle.

Theorem 1.3 (Komlós– Sárközy–Szemerédi [**6**])**.** *For every positive integer k, there exists a positive integer n_0 such that for all positive integers $n > n_0$, any graph G on n vertices with minimum degree $\delta(G) \geq \frac{kn}{k+1}$ contains the k^{th} power of a cycle $C^k_{(k+1)l} \subseteq G$ for any $k+1 \leq (k+1)l \leq n$. If furthermore $K_{k+2} \subseteq G$, then $C^k_l \subseteq G$ for any $k+1 \leq l \leq n$ such that $\chi(C^k_\ell) \leq k+2$.*

A natural question which follows is whether we can determine minimum degree conditions which guarantee the presence of the k^{th} power of paths and cycles of arbitrary given lengths. A reasonable guess is that the answer is characterised by $(k+1)$-partite extremal examples, exemplified by the $k=3$ example in Figure 1a. Allen, Böttcher and Hladký [**1**] established that this is in fact not the case for $k=2$. They answered the question for squared paths and squared cycles, with sharp thresholds attained by a family of extremal graphs which exhibit not a linear dependence between the length of the longest squared path and the minimum degree, but rather piecewise linear dependence with jumps at certain points.



(a) $K_{n-\delta,n-\delta,n-\delta,3\delta-2n}$ **(b)** $G_p(3,n,\delta)$

Figure 1. Graphs for $k=3$

Obtain the n-vertex graph $G_p(k,n,\delta)$ from the disjoint union of $k-1$ independent sets $I_1,\dots,I_{k-1}$ each on $n-\delta$ vertices and r cliques $X_1,\dots,X_r$ with $|X_1| \geq \dots \geq |X_r| \geq |X_1|+1$, by inserting all edges between X_i and I_j for each $(i,j) \in [r] \times [k-1]$ and all edges between I_i and I_j for each $(i,j) \in [k-1]^2$ with $i \neq j$, and taking the maximal value of r for the minimum degree to be at least δ. This is a natural generalisation of the construction in [**1**]. Figure 1b shows an example with $k=3$. Construct the graph $G_c(k,n,\delta)$ in the same way as $G_p(k,n,\delta)$, but also in addition arbitrarily select $v \in X_1$, insert all edges between v and X_i for each $i \in [r]$ such that $|X_i| \neq |X_1|$ and pick the maximal value of r such that the minimum degree is δ. Note that $G_p(k,n,\delta)$ and $G_c(k,n,\delta)$ may not share the same value of r. Define $\mathrm{pp}_k(n,\delta)$ as the length of the longest k^{th} power of a path in $G_p(k,n,\delta)$ and $\mathrm{pc}_k(n,\delta)$ as the length of the longest k^{th} power of a cycle in $G_c(k,n,\delta)$. The behaviour of $\mathrm{pp}_3(n,\delta)$ is illustrated in Figure 2.

Theorem 1.4 (Allen, Böttcher, Hladký [**1**]). *For any $\nu > 0$ there exists an integer n_0 such that for all integers $n > n_0$ and $\delta \in [(\frac{1}{2}+\nu)n, \frac{2}{3}n]$ the following holds for all graphs G on n vertices with minimum degree $\delta(G) \geq \delta$.*

(i) *$P^2_{\mathrm{pp}_2(n,\delta)} \subseteq G$ and $C^2_l \subseteq G$ for every $l \in \mathbb{N}$ with $3 \leq l \leq \mathrm{pc}_2(n,\delta)$ such that 3 divides l.*
(ii) *Either $C^2_l \subseteq G$ for every $l \in \mathbb{N}$ with $3 \leq l \leq \mathrm{pc}_2(n,\delta)$ and $l \neq 5$, or $C^2_l \subseteq G$ for every $l \in \mathbb{N}$ with $3 \leq l \leq 6\delta - 3n - \nu n$ such that 3 divides l.*

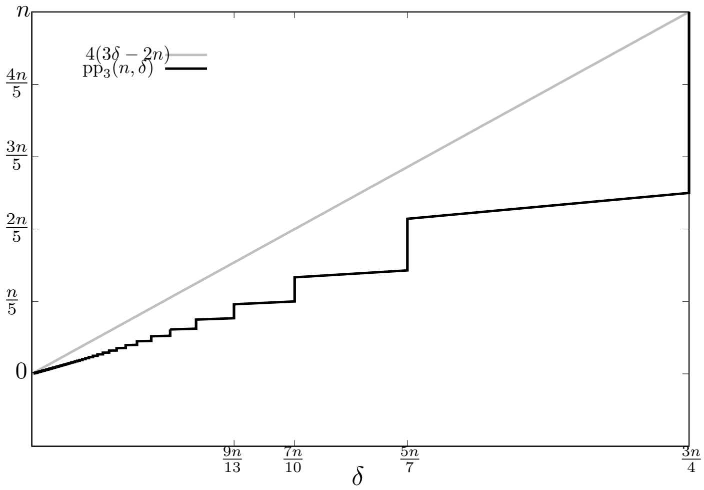


Figure 2. The behaviour of $\mathrm{pp}_3(n,\delta)$

It was conjectured by Allen, Böttcher, and Hladký [**1**] that their result can be naturally generalised to higher powers. Our main theorem states that their conjecture is indeed true.

Theorem 1.5. *Fix $k \geq 3$. For any $\nu > 0$ there exists an integer n_0 such that for all integers $n \geq n_0$ and $\delta \in \left[\left(\frac{k-1}{k}+\nu\right)n, \frac{kn}{k+1}\right]$ the following holds for all graphs G on n vertices with minimum degree $\delta(G) \geq \delta$.*

(i) *$P^k_{\mathrm{pp}_k(n,\delta)} \subseteq G$ and $C^k_\ell \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [k+1, \mathrm{pc}_k(n,\delta)]$ such that $k+1$ divides ℓ.*
(ii) *Either $C^k_\ell \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [k+1, \mathrm{pc}_k(n,\delta)]$ such that $\chi(C^k_\ell) \leq k+2$, or $C^k_\ell \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [k+1, (k+1)(k\delta-(k-1)n)-\nu n]$ such that $k+1$ divides ℓ.*

2. Proof outline

In this section we outline our proof of Theorem 1.5, which uses the well-established technique combining the regularity method and the stability method.

2.1. Regularity method

Szemerédi's regularity lemma [**7**] states that large graphs can be partitioned into finitely many parts such that the edges between almost any pair of parts are evenly distributed. We use a version which accounts for the high minimum degree of the graphs of interest. Given a regularity partition of a graph G we can obtain an auxiliary graph R, termed a *reduced graph* of G, in which the vertex set comprises the vertex classes of the partition and the edge set comprises the regular pairs.

We introduce a notion of connectedness for copies of K_k in a graph. Given a graph G we say that two copies F and F' of K_k are K_{k+1}-*connected* if there exists a sequence of copies of K_k starting with F and ending with F' such that consecutive copies of K_k are part of the same copy of K_{k+1}. This induces an equivalence relation on the copies of K_k in G. Call an equivalence class of this equivalence relation a K_{k+1}-*connected component of* G and a set of vertex-disjoint copies of K_{k+1} which are pairwise K_{k+1}-connected to each other a K_{k+1}-*connected*-K_{k+1}-*factor*. Using standard techniques involving the Blow-up Lemma [**5**] we establish an embedding lemma stating that if we can find a sufficiently large K_{k+1}-connected-K_{k+1}-factor in our reduced graph R then we also have the k^{th} power of paths and cycles of the desired lengths.

2.2. Stability lemma

Our primary new contribution is proving a stability lemma stating that graphs with high minimum degree which do not contain sufficiently large K_{k+1}-connected-K_{k+1}-factors resemble our extremal constructions, i.e. $G_p(k, n, \delta)$ and $G_c(k, n, \delta)$. Denote by $CK_{k+1}F(G)$ the maximum number of vertices covered by a K_{k+1}-connected-K_{k+1}-factor in G.

Lemma 2.1. *Fix $k \geq 3$. Given $\mu > 0$, for any sufficiently small $\eta > 0$ there exists m_0 such that if $\delta \in \left[(\frac{k-1}{k} + \mu)n, \frac{kn-2}{k+1}\right]$ and G is a graph on $n \geq m_0$ vertices with minimum degree $\delta(G) \geq \delta$, then either*

- (C1) *$CK_{k+1}F(G) \geq (k+1)(k\delta - (k-1)n)$, or*
- (C2) *$CK_{k+1}F(G) \geq \mathrm{pp}_k(n, \delta + \eta n)$, or*
- (C3) *G has $k-1$ vertex-disjoint independent sets of combined size at least $(k-1)(n-\delta) - 3k\eta n$ whose removal disconnects G into components which are each of size at most $\frac{19}{10}(k\delta - (k-1)n)$ and for each component X all copies of K_k in G containing at least one vertex of X are K_{k+1}-connected in G.*

Moreover, in (C2) *and* (C3) *each K_{k+1}-component of G contains a copy of K_{k+2}.*

While the lemma statement is analogous to the stability lemma proved in [**1**], the proof is substantially more involved. A key plank of the argument involves showing that vertices which belong to more than one K_{k+1}-connected component induce a K_k-free graph. To this end, we define a family of configurations and prove by induction on k that they are forbidden. Figure 3 and Figure 4 illustrate the configurations for $k = 3$ and $k = 2$ respectively. Note that the $k = 2$ configuration has a $k = 3$ analogue.

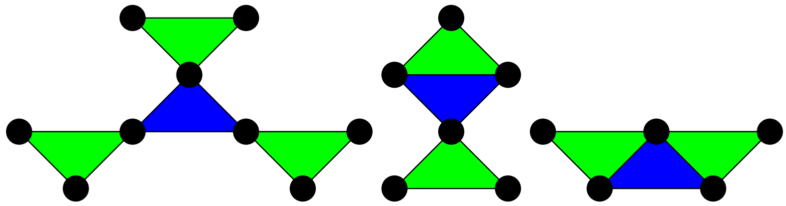


Figure 3. Three configurations with $k = 3$

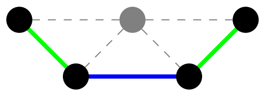


Figure 4. One configuration with $k = 2$

2.3. Sketch proof of Theorem 1.5

Let G be a graph satisfying the hypothesis of Theorem 1.5. The regularity lemma gives a reduced graph R with minimal loss of relative minimum degree. Now apply Lemma 2.1 to R. First consider when we are in cases (C1) and (C2). In these cases, we have large K_{k+1}-connected-K_{k+1}-factors, which allow us to embed the desired k^{th} power paths and cycles.

Otherwise, we must be in case (C3). This means that R resembles our extremal construction, which in turn implies that G must also be similar to our extremal construction. We complete the proof by showing that a graph of this form must contain the k^{th} power of the $\mathrm{pp}_k(n, \delta)$-vertex path and the k^{th} power of cycles of (almost) all lengths up to $\mathrm{pc}_k(n, \delta)$.

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