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ON HEILBRONN TRIANGLE-TYPE PROBLEMS IN HIGHER DIMENSIONS

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ABSTRACT. The Heilbronn triangle problem is a classical geometrical problem that asks for a placement of n points in the unit-square $[0,1]^2$, that maximizes the smallest area of a triangle formed by those points. This problem has natural generalizations to higher dimensions. For integers $k, d \geq 2$ and a set $\mathcal{P}$ of n points in $[0,1]^d$, let $s = \min\{(k-1), d\}$ and $V_k^{(d)}(\mathcal{P})$ be the minimum s-dimensional volume of the convex hull of k points in $\mathcal{P}$. Here, instead of considering the supremum of $V_k^{(d)}(\mathcal{P})$, we consider its average value, $\widetilde{\Delta}_k^{(d)}(n)$, when the n points in $\mathcal{P}$ are chosen independently and uniformly at random in $[0,1]^d$. We prove that $\widetilde{\Delta}_k^{(d)}(n) = \Theta\Big(n^{\frac{-k}{1+|d-k+1|}}\Big)$, for every fixed $k, d \geq 2$.

1. Introduction and main results

Given $n \geq 3$ and a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of n points in $[0,1]^2$, let $A(\mathcal{P})$ be the minimum area of a triangle with all vertices in $\mathcal{P}$. The *Heilbronn triangle problem* asks, for each n, for the supremum of $A(\mathcal{P})$ over all choices of $\mathcal{P}$. We call this value $\Delta_3(n)$.

The exact value of $\Delta_3(n)$ is known only for $n \leq 7$, and the problem is still wide open for all $n > 7$. This problem has a rich history (see **[5, 6, 16]** for some optimal configurations and constructive lower bounds). For general n, a trivial upper bound, given by splitting the square into squares of side length $\sqrt{3/n}$ and using pigeonhole principle, is $\Delta_3(n) \leq 3/(2n)$. Erdős established the lower bound $\Delta_3(n) = \Omega(1/n^2)$, while Roth **[14]** and Schmidt **[15]** found upper bounds on $\Delta_3(n)$. For large n, the best known lower and upper bounds are by Komlós, Pintz and Szemerédi **[10, 11]**, for constants $c_1, c_2 > 0$:

$$c_1 \frac{\log n}{n^2} \leq \Delta_3(n) \leq \frac{2^{c_2\sqrt{\log n}}}{n^{\frac{8}{7}}}.$$

A generalization of this problem has been considered by Schmidt **[15]** for integers $n \geq k \geq 3$. For a set $\mathcal{P}$ of n points in $[0,1]^2$, let $A_k(\mathcal{P})$ be the minimum area of the convex hull of k distinct points in $\mathcal{P}$, and let $\Delta_k(n)$ be the supremum of $A_k(\mathcal{P})$. For fixed $k \geq 3$, the currently best known lower bound is

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$\Delta_k(n) = \Omega((\log n)^{1/(k-2)}/n^{(k-1)/(k-2)})$, see [12]. However, for fixed $k \geq 4$, only the (trivial) upper bound $\Delta_k(n) = O(1/n)$ is known.

An extension to dimension d, for $d \geq 3$, was also considered by Barequet and Naor [1, 2]. The $(k-1)$-dimensional volume of the convex hull of k points $P_1, \ldots, P_k \in [0,1]^d$, $2 \leq k \leq d+1$, is given by

$$V_k^{(d)}(P_1, \ldots, P_k) := \frac{1}{(k-1)!} \cdot \prod_{i=2}^{k} \operatorname{dist}(P_i; [P_1, \ldots, P_{i-1}]),$$

where $\operatorname{dist}(P_i; [P_1, \ldots, P_{i-1}])$ is the Euclidean distance of P_i to the affine space $[P_1, \ldots, P_{i-1}]$. For $k > d+1$ we compute the d-dimensional volume by splitting the convex hull of $P_1, \ldots, P_k$ into interior disjoint d-simplices.

Given k, d and a placement $\mathcal{P}$ of n points in $[0,1]^d$, let $s = \min\{(k-1), d\}$ and let $V_k^{(d)}(\mathcal{P})$ be the minimum s-dimensional volume of the convex hull of k distinct points in $\mathcal{P}$, and let $\Delta_k^{(d)}(n)$ be the supremum of $V_k^{(d)}(\mathcal{P})$ over all choices of $\mathcal{P}$ with $|\mathcal{P}| = n$. For fixed d and k, where $3 \leq k \leq d+1$, the best known lower bound for $\Delta_k^{(d)}$ is $\Delta_k^{(d)}(n) = \Omega((\log n)^{1/(d-k+2)}/n^{(k-1)/(d-k+2)})$, see [13].

In connection with range searching problems, Chazelle [4] investigated $\Delta_k^{(d)}(n)$ when k is a function of n. He showed that, in any fixed dimension $d \geq 2$, for $\log n \leq k \leq n$, we have the asymptotically correct order $\Theta(k/n)$ for $\Delta_k^{(d)}(n)$.

These problems have proved to be remarkably difficult, and it was natural to address a simpler problem, namely to determine the average value of $V_k^{(d)}(\mathcal{P})$ when each point in a placement $\mathcal{P} \subset [0,1]^d$ of n points is chosen independently and uniformly at random, denoted $\widetilde{\Delta}_k^{(d)}(n)$. Note this is also well-defined for $d = 1$ (there is a well-known short proof for $\widetilde{\Delta}_2^{(1)}(n) = 1/(n-1)^2$).

Jiang, Li and Vitány [9] showed that $\widetilde{\Delta}_3^{(2)}(n) = \Theta(1/n^3)$ using Kolmogorov complexity. Grimmett and Janson [7] strengthened this to $\lim_{n\to\infty}(n^3 \cdot \widetilde{\Delta}_3^{(2)}(n)) = 1/2$, and also determined the analogous limit when the n points are chosen with more general probability distributions. They also found the asymptotic probability distribution of $A_3(\mathcal{P})$ (and, more generally, of the size of the ℓ-th smallest triangle).

In our work, we determine the order of $\widetilde{\Delta}_k^{(d)}(n)$ for every fixed d and k.

Theorem 1.1. *Let $d, k \geq 2$ be fixed integers. There exist positive constants $c_{d,k}$ and $C_{d,k}$ such that, for n sufficiently large, it is*

$$\frac{c_{d,k}}{n^{\frac{k}{1+|d-k+1|}}} \leq \widetilde{\Delta}_k^{(d)}(n) \leq \frac{C_{d,k}}{n^{\frac{k}{1+|d-k+1|}}}.$$

In this note, we prove some cases of Theorem 1.1 to illustrate how the full argument goes. One can also show a discretized version, i.e., a d-dimensional $K \times \cdots \times K$-grid is embedded on $[0,1]^d$ and points are placed on the grid-points, where K is sufficiently large in terms of n. The arguments are rather similar and integrals become sums.

2. Areas of triangles in $[0,1]^2$

We first give a detailed argument for Theorem 1.1 in the case $k = 3$ and $d = 2$ and then briefly sketch the argument for fixed $k \geq 3$ and $d = 2$, whose ideas can be generalized to any fixed $k, d \geq 3$. Note that the case $k = 3$ and $d = 2$ was already solved in the literature **[7, 9]**, but our proof is very short and works as a model for the other cases. Let $\operatorname{dist}(P, Q)$ denote the Euclidean distance between the points P and Q.

Proposition 2.1. *Let P_1, P_2, P_3 be points selected independently and uniformly at random from $[0,1]^2$. Let T be the triangle $P_1P_2P_3$. Then, for every $0 \leq A \leq 1$,*

$$A \leq \mathbf{P}(\operatorname{area}(T) \leq A) \leq 12A.$$

Sketch. For the lower bound, suppose that the first two points P_1, P_2 have been selected. If there is no point Q in $[0,1]^2$ such that the triangle P_1P_2Q has area larger than A, then the probability that the area of T is at most A is 1. Otherwise, there is a point Q in $[0,1]^2$ such that the triangle P_1P_2Q has area A. The probability that the area of T is at most A is at least the probability that P_3 lies in the triangle P_1P_2Q, which is equal to A.

For the upper bound, for each $i, j \in \{1, 2, 3\}$, consider the case where P_iP_j is the longest side of the triangle and use the union bound. In each case, the third point is contained in a rectangle of area $4A$. □

Lemma 2.2. *For any $n \geq 3$, we have $\widetilde{\Delta}_3^{(2)}(n) \geq 1/(8n^3)$.*

Proof. Place n points independently and uniformly at random in the unit-square $[0,1]^2$. We set $A = 1/(4n^3)$. By Proposition 2.1 and the union bound, the probability that at least one of the triangles with vertices among the n points has area at most A is at most $12A \cdot \binom{n}{3} \leq 1/2$. Then, by Markov's inequality, the expected area $\widetilde{\Delta}_3^{(2)}(n)$ of a triangle of minimum area is at least $1/(8n^3)$. □

For the upper bound, we will use the following Suen-type correlation inequality (see Theorem 1 in **[8]**). For distinct events B_1, B_2, B_3 in a probability space, $B_1 \sim B_2$ denotes that B_1 and B_2 are dependent, and $B_1 \sim \{B_2, B_3\}$ denotes that B_1 is not mutually independent of the set $\{B_2, B_3\}$, that is, it is dependent on B_2 or B_3 or $B_2 \cap B_3$.

Theorem 2.3. *Let $B_1, \ldots, B_k$ be distinct events in a given probability space. Let $M = \prod_{i=1}^{k} \mathbf{P}(\overline{B_i})$ and $D = \sum_{B_i \sim B_j} \mathbf{P}(B_i \wedge B_j)$. Assume that for every pair of distinct dependent events $B_i \sim B_j$ the number of events B_g with $B_g \sim \{B_i, B_j\}$ is at most α and that $\mathbf{P}(B_i) \leq \varepsilon$ for every $i \in \{1, \ldots, k\}$. Then,*

$$\mathbf{P}\left(\wedge_{i=1}^{k} \overline{B_i}\right) \leq M \cdot \mathrm{e}^{\frac{D}{(1-\varepsilon)^{\alpha}}}.$$

Lemma 2.4. *For all sufficiently large n, we have $\widetilde{\Delta}_3^{(2)}(n) \leq 18/n^3$.*

Proof. Place n points independently and uniformly at random in $[0,1]^2$. For each set I of three of those points, T_I is the triangle with vertices in I.

We shall give an upper bound on the probability that *all* triangles have 'large' area. Fix $0 < A \leq 1$. For each $x \in \{1, \ldots, \lfloor 2\ln n \rfloor\}$, let $B_I^{(x)}$ denote the event "area$(T_I) \leq Cx/n^3$", for a suitable constant $C > 0$. By Proposition 2.1, we have

$$\frac{Cx}{n^3} \leq \mathbf{P}(B_I^{(x)}) \leq \frac{12Cx}{n^3} = \varepsilon(x).$$

For $I \neq J$, the events $B_I^{(x)}$ and $B_J^{(x)}$ are dependent only if the triangles T_I and T_J have exactly one vertex or one side in common. In the first case, let $I = \{P_1, P_2, P_3\}$ and $J = \{P_1, Q_1, Q_2\}$. Without loss of generality, place P_1, P_2, P_3, Q_1, Q_2 in $[0,1]^2$ in this order. The event $B_I^{(x)}$ happens with probability at most $12Cx/n^3$. Regardless of P_1's position, for each $z \in [0, \sqrt{2}]$ (where $\sqrt{2}$ is the longest possible distance between points in $[0,1]^2$), the probability that $\operatorname{dist}(P_1, Q_1)$ is in the infinitesimal interval $[z, z + \mathrm{d}z]$ is at most $2\pi z\,\mathrm{d}z$ (i.e., the area of the appropriate annulus). For $B_J^{(x)}$ to hold, the last point, Q_2, is contained in a rectangle of area at most $4\sqrt{2}\,Cx/(zn^3)$.

In the second case, denoting the length of the common side PP' of T_I and T_J by y, place one endpoint of the common side anywhere in $[0,1]^2$; the other endpoint, P', satisfies that $\operatorname{dist}(P, P')$ is in the infinitesimal interval $[y, y + \mathrm{d}y]$ with probability at most $2\pi y\,\mathrm{d}y$, and the two remaining vertices of T_I and T_J must be contained in a rectangle of area at most $\min\{1, 4\sqrt{2}C/(yn^3)\}$. For n sufficiently large, we conclude that

$$\begin{aligned} D^{(x)} &= \sum_{B_I^{(x)} \sim B_J^{(x)}} \mathbf{P}(B_I^{(x)} \wedge B_J^{(x)}) \\ &\leq \binom{n}{5} \frac{12Cx}{n^3} \int_0^{\sqrt{2}} \frac{4\sqrt{2}\,Cx}{zn^3}\, 2\pi z\,\mathrm{d}z + \binom{n}{4} \int_0^{\sqrt{2}} \left(\min\left\{1, \frac{4\sqrt{2}\,Cx}{yn^3}\right\}\right)^2 2\pi y\,\mathrm{d}y \\ &\leq \frac{8\pi C^2 x^2}{5n} + \frac{192\pi C^2 x^2 \ln n}{n^2} \leq \frac{1.7\pi C^2 x^2}{n}. \end{aligned}$$

Moreover, letting $\binom{[n]}{3}$ be the set of all subsets of three points, by Proposition 2.1 we have

$$M^{(x)} = \prod_{I \in \binom{[n]}{3}} \mathbf{P}\left(\overline{B_I^{(x)}}\right) \leq \left(1 - \frac{Cx}{n^3}\right)^{\binom{n}{3}}.$$

Now, use Theorem 2.3. Clearly, $\max_{B_I \sim B_J} |\{G \in \binom{[n]}{3} : B_G \sim \{B_I, B_J\}\}| \leq 3n^2$, for $n \geq 15$. Setting $\alpha = 3n^2$, using that $1 + z \leq \mathrm{e}^z$, for all z, and $x \leq 2\ln n$, we infer that, for n large,

$$\begin{aligned} \mathbf{P}(\wedge_{I \in \binom{[n]}{3}} \overline{B_I^{(x)}}) &\leq M^{(x)} \cdot \mathrm{e}^{\frac{D^{(x)}}{(1-\varepsilon(x))^\alpha}} \leq \left(1 - \frac{Cx}{n^3}\right)^{\binom{n}{3}} \cdot \exp\left(\frac{1.7\pi C^2 x^2/n}{\left(1 - \frac{12Cx}{n^3}\right)^{3n^2}}\right) \\ &\leq \exp\left(\frac{-Cx}{n^3}\frac{(n-2)^3}{6}\right) \cdot \exp\left(\frac{2\pi C^2 x^2}{n}\right) \leq \exp\left(-\frac{Cx}{7}\right). \qquad (1) \end{aligned}$$

Therefore, letting $C = 7$, the probability that the minimum area of a triangle is larger than $7x/n^3$ is at most e^{-x}. In particular, the probability that such area is in the range $[7x/n^3, 7(x+1)/n^3]$ is also at most e^{-x}.

By the result of Komlós, Pintz and Szemerédi [**10**] mentioned before, for a constant $c > 0$ and n sufficiently large, in any placement of n points in the unit-square $[0,1]^2$ the minimum area of a triangle is at most $2^{c\sqrt{\log n}}/n^{8/7}$.

Thus, for n large, the average minimum area $\widetilde{\Delta}_3^{(2)}(n)$ of a triangle satisfies

$$\begin{aligned}\widetilde{\Delta}_3^{(2)}(n) &\le \sum_{x=0}^{\lfloor 2\ln n\rfloor} \frac{1}{\mathrm{e}^x}\cdot\frac{7(x+1)}{n^3} + \mathrm{e}^{-2\ln n}\cdot\frac{2^{c\sqrt{\log n}}}{n^{\frac{8}{7}}} \\ &= \sum_{x=0}^{\lfloor 2\ln n\rfloor} \frac{1}{\mathrm{e}^x}\cdot\frac{7(x+1)}{n^3} + \frac{1}{n^2}\cdot\frac{2^{c\sqrt{\log n}}}{n^{\frac{8}{7}}} \le \frac{18}{n^3},\end{aligned}$$

where we used $\sum_{x=0}^{\infty} 7(x+1)/\mathrm{e}^x = 7e^2/(\mathrm{e}-1)^2 \le 17.6 < 18$. □

Remark: If instead of the upper bound $2^{c\sqrt{\log n}}/n^{8/7}$ we had used only the trivial upper bound $\Delta_3(n) \le 3/(2n)$, we would obtain $(19.1)/n^3$ instead of $18/n^3$ in the above result.

3. Areas of convex hulls of k points in $[0,1]^2$

The goal of this section is to discuss briefly the bounds on $\widetilde{\Delta}_k^{(2)}(n)$, when $k \ge 4$ is fixed. First, we generalize Lemma 2.2.

Lemma 3.1. *Let $k \ge 3$ be fixed. For n sufficiently large, we have*

$$\widetilde{\Delta}_k^{(2)}(n) \ge \frac{((k-2)!)^{1/(k-2)}}{8\, n^{k/(k-2)}},$$

Proof. Place n points independently and uniformly at random in the unit-square $[0,1]^2$. For each set J of k of those points, let K_J be their convex hull. Clearly, if K_J has area at most B, then every triangle $P_gP_hP_i$ with $\{P_g, P_h, P_i\} \subseteq J$ has area at most B. Fix some J and assume without loss of generality that $J = \{P_1, \ldots, P_k\}$.

Let S be the event "area$(K_J) \le B$" and $E_{g,h}$ be the event "the distance between the points P_g and P_h is at least as large as the distance between the other pairs of points in the set J". By the union bound, we have

$$\mathbf{P}(S) \le \sum_{1\le g<h\le k} \mathbf{P}(S \cap E_{g,h}). \tag{2}$$

To compute $\mathbf{P}(S\cap E_{g,h})$, note that for any choice of P_g and P_h any other point P_t from J must lie in a rectangle of area at most $4B$. Since these points are chosen independently, we have $\mathbf{P}(S \cap E_{g,h}) \le (4B)^{k-2}$, hence, expression (2) is at most $\binom{k}{2}(4B)^{k-2}$.

If we set $B = c/n^{k/(k-2)}$, by the union bound, the probability that at least one of the convex hulls K_J has area at most B is bounded above by

$$\binom{n}{k}\binom{k}{2}(4B)^{k-2} \leq \frac{(4c)^{k-2}}{2\cdot(k-2)!}.$$

In particular, if we choose $c = (1/4)\big((k-2)!\big)^{1/(k-2)}$, the above upper bound is $1/2$, and by Markov's inequality, we have

$$\widetilde{\Delta}_k^{(2)}(n) \geq \left(1 - \frac{(4c)^{k-2}}{2\cdot(k-2)!}\right)\cdot B = \frac{1}{2}\cdot\frac{c}{n^{\frac{k}{k-2}}} = \frac{((k-2)!)^{\frac{1}{k-2}}}{8n^{\frac{k}{k-2}}}. \qquad \square$$

For a configuration of k points $P_1, \ldots, P_k$ in $[0,1]^2$, we define P_i and P_j to be *extremal points* if their Euclidean distance is at least as large as the distance of any pair of the points $P_1, \ldots, P_k$.

Lemma 3.2. *Let $k \geq 3$ be fixed. There is an absolute constant $C_k > 0$ such that, for n sufficiently large, we have $\widetilde{\Delta}_k^{(2)}(n) \leq C_k/n^{k/(k-2)}$.*

Proof. Place n points independently and uniformly at random in the unit-square $[0,1]^2$. For each subset I of k points, let K_I be the convex hull of points in I. We give an upper bound on the probability that every K_I has 'large' area. By our previous result, we know that $\mathbf{P}(\text{area}(K_I) \leq B) \leq \binom{k}{2}(4B)^{k-2} \leq k^2(4B)^{k-2}$. Moreover, by Proposition 2.1, we have $\mathbf{P}(\text{area}(K_I) \leq B) \geq B^{k-2}$, for $B \leq 1$.

Let $B_I^{(x)}$ denote the event "area$(K_I) \leq B(x)$", where $B(x) = Cx/n^{k/(k-2)}$ for a suitable constant $C > 0$, $x \in \{1, \ldots, \lfloor (2\ln n)^{1/(k-2)} \rfloor\}$ and a subset I containing k of the n points. We have

$$\frac{(Cx)^{k-2}}{n^k} \leq \mathbf{P}(B_I^{(x)}) \leq \frac{k^2 4^{k-2}(Cx)^{k-2}}{n^k} = \varepsilon(x).$$

Fix distinct k-element sets I and J of points. The events $B_I^{(x)}$ and $B_J^{(x)}$ are dependent only if I and J intersect non-empty. Let $\ell = |I \cap J|$, and assume $1 \leq \ell \leq k-1$. Let $I = \{P_1, \ldots, P_k\}$, and $J = \{P_1, \ldots, P_\ell, Q_{\ell+1}, \ldots, Q_k\}$. We estimate the probability $\mathbf{P}(B_I^{(x)} \wedge B_J^{(x)})$.

First let $\ell = 1$, thus $|I \cap J| = 1$. Given P_1, there are two possibilities for the convex hulls of K_I and K_J, respectively: (i) P_1 is an extremal point for K_I or K_J, or (ii) P_1 is not an extremal point for K_I nor K_J.

Adding the two sub-cases, for $\ell = 1$, we can show for constants $C', C'' > 0$ that

$$\begin{aligned} D_1^{(x)} &= \sum_{B_I^{(x)} \sim B_J^{(x)};\, |I\cap J|=1} \mathbf{P}(B_I^{(x)} \wedge B_J^{(x)}) \\ &\leq C'(4B(x))^{2k-4} n^{2k-1} \leq \frac{C''(Cx)^{2k-4}}{n}. \end{aligned} \tag{3}$$

Next let K_I and K_J have exactly ℓ points in common, $2 \leq \ell \leq k-1$. Given the points $P_1, \ldots, P_\ell$, there are three possibilities for the convex hulls of the points $P_1, \ldots, P_k$ and $P_1, \ldots, P_\ell, Q_{\ell+1}, \ldots, Q_k$, respectively: (i) two of the common points $P_1, \ldots, P_\ell$ are extremal for K_I or K_J; (ii) case (i) does not hold and exactly one of the points $P_1, \ldots, P_\ell$ is extremal for K_I or K_J; (iii) none of the points $P_1, \ldots, P_\ell$ is extremal for K_I or K_J.

The largest upper bound for the probability that the convex hulls of K_I and K_J have area at most $B(x)$ arises in case (iii) and is at most

$$C'' \cdot B(x)^{2k-\ell-2} \cdot \ln^3 n, \tag{4}$$

for a constant $C'' > 0$. This bound comes from multiplying $C'(4B(x))^{2k-\ell-4}$ by

$$\int_0^{\sqrt{2}} \sum_{\substack{t=-\lceil\sqrt{2}/y\rceil-1;\\ t\neq 0}}^{\lceil\sqrt{2}/y\rceil+1} \min\left\{1, \frac{4\sqrt{2}B(x)}{|t|y}\right\} \sum_{\substack{s=-\lceil\sqrt{2}/y\rceil-1;\\ s\neq 0}}^{\lceil\sqrt{2}/y\rceil+1} \min\left\{1, \frac{4\sqrt{2}B(x)}{|s|y}\right\} \cdot 2\pi y \, \mathrm{d}y.$$

By (4) and because $B(x) = Cx/n^{k/(k-2)}$, we infer that for $\ell \geq 2$

$$D_\ell^{(x)} = \sum_{B_I^{(x)} \sim B_J^{(x)};\, |I\cap J|=\ell} \mathbf{P}(B_I^{(x)} \wedge B_J^{(x)}) \leq C'' \cdot \frac{(Cx)^{2k-\ell-2}}{n^{\frac{2k-2\ell}{k-2}}} \cdot \ln^3 n. \tag{5}$$

From equations (3) and (5), together with $x \leq (2\ln n)^{1/(k-2)}$, $k \geq 4$, and n large, we have a constant $C''' > 0$ such that

$$D^{(x)} = \sum_{B_I^{(x)} \sim B_J^{(x)}} \mathbf{P}(B_I^{(x)} \wedge B_J^{(x)}) = \sum_{\ell=1}^{k-1} D_\ell^{(x)} \leq \frac{C''' x^{2k-4} \cdot \ln^3 n}{n^{\frac{2}{k-2}}} \leq \frac{4C''' \ln^5 n}{n^{\frac{2}{k-2}}}.$$

Moreover, using $1 + z \leq e^z$ for all z, for n large,

$$M^{(x)} = \prod_{I \in \binom{[n]}{k}} \mathbf{P}(\overline{B_I^{(x)}}) \leq \left(1 - \frac{(Cx)^{k-2}}{n^k}\right)^{\binom{n}{k}} \leq \mathrm{e}^{-\frac{(Cx)^{k-2}}{n^k}\binom{n}{k}} \leq \mathrm{e}^{-\frac{(Cx)^{k-2}}{2\,k!}}.$$

We have $\max_{B_I^{(x)} \sim B_J^{(x)}} |\{G \in \binom{[n]}{k} \colon B_G^{(x)} \sim \{B_I^{(x)}, B_J^{(x)}\}\}| \leq 3k^2 n^{k-1} = \alpha$. For n sufficiently large, $(1-\varepsilon(x))^\alpha \geq 1/2$. Hence, by $x \leq (2\ln n)^{1/(k-2)}$ and Theorem 2.3,

$$\begin{aligned}
\mathbf{P}(\wedge_{I\in\binom{[n]}{k}} \overline{B_I^{(x)}}) &\leq M^{(x)} \cdot \exp\left(\frac{4C''' \cdot \ln^5 n}{n^{\frac{2}{k-2}}(1-\varepsilon(x))^\alpha}\right) \\
&\leq \exp\left(-\frac{(Cx)^{k-2}}{2\,k!}\right) \cdot \exp\left(\frac{8C''' \cdot \ln^5 n}{n^{\frac{2}{k-2}}}\right) \\
&\leq \exp\left(\frac{-(Cx)^{k-2}}{3 \cdot k!}\right).
\end{aligned}$$

Then, for $C = (3 \cdot k!)^{1/(k-2)}$, the probability that the minimum area of the convex hull of k points is within the range $[Cx/n^{k/(k-2)}, C(x+1)/n^{k/(k-2)}]$ is at most $\mathrm{e}^{-x^{k-2}}$. Splitting the unit-square $[0,1]^2$ into squares of side length $\sqrt{k/n}$ and using the pigeonhole principle one shows that $\Delta_k^{(2)}(n) \leq k/n$. Therefore we have

$$\widetilde{\Delta}_k^{(2)}(n) \leq \sum_{x=0}^{\lfloor (2\ln n)^{\frac{1}{k-2}} \rfloor} \mathrm{e}^{-x^{k-2}} \cdot \frac{C(x+1)}{n^{\frac{k}{k-2}}} + \mathrm{e}^{-2\ln n} \cdot \frac{k}{n} = O\left(\frac{1}{n^{\frac{k}{k-2}}}\right). \qquad \square$$

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