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RECENT DEVELOPMENTS ON UNAVOIDABLE PATTERNS IN 2-COLORINGS OF THE COMPLETE GRAPH

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Abstract. In this manuscript, we review recent developments concerning unavoidable patterns in 2-edge colorings of the complete graph.

1. Introduction

Given a graph G, the 2*-color Ramsey number of* G is the minimum integer $R(G)$ such that, for every $n \geq R(G)$, there is a monochromatic copy of G in any 2-coloring of $E(K_n)$. The existence of the Ramsey number $R(G)$ is guaranteed by Ramsey's Theorem [**9**], which states that, for sufficiently large n, every 2-coloring of the complete graph on n vertices contains a monochromatic clique on k vertices. Erdős and Szekeres showed that $R(k) = R(K_k) < 2^{2k}$ and Erdős gave the lower bound $R(k) > 2^{k/2}$ for $k > 2$. There have been several improvements of these bounds but the constant factors of the exponents remain the same. For an overview on results about Ramsey theory in graphs, see the survey [**5**].

To force the existence of graphs in other color patterns, we need, as a natural minimum requirement, not only to ensure a large n, but also a minimum amount of edges of each color. A 2-edge-colored complete graph K_{2t} is said to be of *type A* if the edges of one of the colors induce a complete graph K_t, and it is of *type B* if the edges of one of the colors induce two disjoint complete graphs K_t. It has been shown in [**3**] that, in both coloring types, there are infinitely many t's for which such colorings have precisely half of the edges on each of the colors. It is clear that every complete graph K_k contained as a subgraph of a type A or a type B colored K_n will have essentialy the same color pattern as the hostgraph except for maybe the proportion of colors among the edges. This shows that these 2-edge-colorings are the only types of patterns that are possibly unavoidable in 2-edge-colorings that are far from being monochromatic. Generalizing the classical Ramsey problem, Bollobás (see [**6**]) conjectured that for every $\varepsilon > 0$ and every positive integer k, there is an n_0 such that, for $n \geq n_0$, every 2-edge-coloring of

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K_n with at least $\varepsilon\binom{n}{2}$ edges in each color contains a type A or a type B colored K_k. This conjecture was proved affirmative by Cutler and Montágh in 2008 [**6**]. In this paper, we discuss the different results that have emerged from or in parallel to Bollobás' conjecture which concern improvements, generalizations and similar approaches of this problem. In particular, we make emphasis on the approach engaged in [**2, 3, 4**] by the authors of this manuscript, where we prescind from the ε-density on the coloring of the edges of K_n to substitute it with a less strong condition that will let us define another parameter $\varphi(n,k)$, which we show to be $o(n^2)$ [**2**], that will represent the maximum number of edges of one of the colors for which there is a 2-edge-coloring of K_n without a type-A or a type-B colored K_k. This gives the problem also a Turán flavor in the sense that, more generally for a graph G, we are interested in finding the maximum edge number that can have the smallest color class in a 2-coloring of $E(K_n)$ which is free of a copy of G in the prescribed pattern, as well as in characterizing the extremal colorings.

2. Unavoidable patterns in ε-balanced colorings

For $0 < \varepsilon < \frac{n}{2}$, we say that a 2-edge-coloring of the complete graph K_n is *ε-balanced*, if it contains at least $\varepsilon\binom{n}{2}$ edges in each color. As mentioned in the introduction, Bollobás raised the following conjecture.

Conjecture 2.1 (Bollobás, see [**6**]). *For every $0 < \varepsilon < \frac{1}{2}$ and every positive integer k, there is an integer $n(k,\varepsilon)$ such that every ε-balanced 2-edge-coloring of K_n with $n \geq n(k,\varepsilon)$ contains a type-A or a type-B colored K_k.*

Cutler and Montágh [**6**] proved the conjecture in 2008. They gave a quite complicated proof relying on probabilistic arguments and yielding an upper bound of $n(k,\varepsilon) < 4^{k/\varepsilon}$. Fox and Sudakov [**7**] gave a much simpler proof of an upper bound that is better than Cutler and Montágh's for small ε. They also showed by means of a simple probabilistic argument that this bound is tight up to a constant factor in the exponent for all k and ε.

Theorem 2.2 ([**7**]). *For every $0 < \varepsilon < \frac{1}{2}$, if $n \geq (16/\varepsilon)^{2k+1}$, then every ε-balanced 2-edge-coloring of K_n contains a K_k of type A or of type B.*

In their paper, Fox and Sudakov [**7**] also give analogous results for tournaments.

Observe that, if we consider, more generally, a graph G with a certain 2-edge coloring (instead of a complete graph K_k), then Theorem 2.2 already characterizes those graphs G for which, for n sufficiently large, every ε-balanced 2-edge-coloring of K_n contains a color-consistent copy of G. Namely, setting $n(G) = k$, if $n \geq n(k,\varepsilon)$ and the colored copy of G can be found as a colored-consistent subgraph of both, a type-A-colored K_k and a type-B-colored K_k, then it will be contained in every ε-balanced 2-edge-coloring of K_n. If this is not the case, then there will be infinitely many n's for which there is a $\frac{1}{2}$-balanced type-A-coloring of K_n or a type-B-coloring of K_n [**3**] that will not contain a color-consistent copy of G.

Recently, Bowen, Lamaison and Müyesser [**1**] gave and proved a generalization of Bollobás' conjecture to $q \geq 2$ colors. To do this, they define, for $0 < \varepsilon \leq \frac{n}{q}$, the so-called *$\varepsilon$-balanced q-Ramsey number* $R_\varepsilon^q(G)$ for a graph G equipped with certain q-edge-coloring: $R_\varepsilon^q(G)$ is the minimum integer N such that, for $n \geq N$, every ε-balanced q-edge coloring of K_n contains a color-consistent copy of G. Here, analogously to the case $q = 2$, an *ε-balanced q-edge coloring* means that there are at least $\varepsilon\binom{n}{2}$ edges in each color. If no such N exists, we say that $R_\varepsilon^q(G) = \infty$. For a family $\mathcal{F}$ of q-edge-colored graphs, we define $R_\varepsilon^q(\mathcal{F})$ as the minimum integer N such that, for $n \geq N$, every q-edge coloring of K_n contains a color-consistent copy of some $G \in \mathcal{F}$. Hence, if k is an even integer and $\mathcal{F}_k$ is the family containing a type-A and a type-B colored K_k, then Theorem 2.2 is equivalent to the following.

Theorem 2.3 ([**7**])**.** *For every $0 < \varepsilon < \frac{1}{2}$, $R_\varepsilon^2(\mathcal{F}_k) \leq (16/\varepsilon)^{2k+1}$.*

The q-color version in [**1**] includes a generalization of the type-A and type-B colorings of K_k to a family $\mathcal{F}_k^q$ of q-edge-colored K_k's. Similarly to the case $q = 2$, this completely characterizes the q-colored graphs G for which $R_\varepsilon^q(G) < \infty$. Moreover, the following generalized version of Theorem 2.2 is given in [**1**].

Theorem 2.4 ([**1**])**.** *For every integer $q \geq 2$ and every $0 < \varepsilon < \frac{1}{q}$, there is a positive constant $c := c(q)$ such that $R_\varepsilon^q(\mathcal{F}_k^q) \leq \varepsilon^{-ck}$.*

The bound of Theorem 2.4 is asymptotically tight by a simple probabilistic construction [**1**]. In the same paper, the authors also consider the family $M_{k,l}$ of certain asymmetric colored graphs G (that appear as color-consistent copies in any type-A and type-B colored $K_{n(G)}$) and show that, for a fixed k, there is a constant $C := C(\varepsilon, l)$ such that $R_\varepsilon^2(M_{l,k}) \leq C \cdot R(k)$, where $R(k)$ is the the classical Ramsey number given in the introduction of this paper.

3. Unavoidable patterns in colorings of the complete graph with at least $o(n^2)$ edges of each color

In this section, we will review the results of [**2**], where we prescind from the ε-density on the coloring of the edges of K_n to talk about a precise minimum number of edges in each of the colors, which will prove to be subquadratic in n. The results in the afore mentioned paper where obtained independently from [**6**], [**7**] and [**1**] as we were foccussing mostly on the minimum number of edges in a 2-edge coloring of K_n required to guaratee the existence of a color consistent copy of a graph G with certain color-pattern.

To be more specific, given a graph G with $e(G)$ edges, non-negative integers r and b such that $r + b = e(G)$, and a 2-coloring $f\colon E(K_n) \to \{red, blue\}$, we say that f induces an *(r, b)-colored copy* of G, if there is a copy of G in K_n such that f assigns the color red to exactly r edges and the color blue to exactly b edges of that copy of G. We say that G is *r-tonal* if, for n large enough, there is an integer $\varphi_r(n, G)$ such that every 2-coloring $f\colon E(K_n) \to \{red, blue\}$ with more than $\varphi_r(n, G)$ edges in each color contains either an $(r, e(G) - r)$-colored copy or an $(e(G) - r, r)$-colored copy of G. In particular, we are interested in $e(G)/2$-tonal graphs, which we call *balanceable graphs*, and in graphs G that are r-tonal for every $0 \leq r \leq e(G)$, which we call *omnitonal graphs*. The study of these graph families led us to its characterization via a completely analogous theorem to Theorem 2.2 or 2.3 but without the ε-balanced coloring. Our proof relies on the classical Ramsey Theorem [**9**] as well as on the well-known Kővari-Sós-Turán theorem [**8**] that yields an $o(n^2)$-bound on the extremal number for bipartite graphs and, particularly,

$$\mathrm{ex}(n, K_{t,t}) = \mathcal{O}(n^{2-1/t}), \tag{1}$$

where $\mathrm{ex}(n, G)$ stands for the *extremal number of* G, that is, the maximum number of edges in a graph with n vertices containing no copy of G. The Kővari-Sós-Turán theorem yields also

$$z(n, t) = \mathcal{O}(n^{2-1/t}), \tag{2}$$

where $z(n, t)$ is the *Zarankiewicz number*, that is, the maximum number of edges in a bipartite graph with n vertices in each part, containing no copy of $K_{t,t}$.

Theorem 3.1 ([**2**]). *Let k be a positive even integer. Then there are positive integers $N := N(k)$, $m = m(k)$ and $\varphi(n, k) = \mathcal{O}(n^{2-\frac{1}{m}})$ such that, for every $n \geq N$, every 2-edge-coloring of K_n with at least $\varphi(n, k)$ edges from each color contains a type-A or a type-B colored copy of K_k.*

Proof. (Sketch) Let q be an integer large enough to satisfy $z(2q, k/2) \leq 2q^2$. This is possible because of (2). Let $m = R(q)$ and define

$$\varphi(n, k) = \mathrm{ex}(n, K_{m,m}) + m(m-1) + 2m(n-2m) + 1.$$

Take an n large enough to fulfill $2\varphi(n, k) \leq \binom{n}{2}$. This is possible because of (1). By the definition of the extremal number $ex(n, K_{m,m})$, there is a blue and a red $K_{m,m}$. By the definition of m and the Ramsey number $R(q)$, in each partition set of the blue and the red $K_{m,m}$, there is a monochromatic K_q. Analyzing the different possibilities of combinations of red and/or blue K_q's, one obtains either a type-A-colored K_{2q} or a type-B-colored K_{2q}, in which case we finish easily, or we get a blue and a red colored K_{2q}. Considering, in the latter case, the edges that share these vertex disjoint red and blue cliques, we can finish by means of the Zarankiewicz number $z(2q, k/2)$. □

As in the ε-balanced case, Theorem 3.1 yields a characterization of r-tonal graphs. Namely, setting $n(G) = k$, if $n \geq N(k)$ and the colored copy of G can be found as a colored-consistent subgraph of both, a type-A-colored K_k and a type-B-colored K_k, then it will be contained in every 2-edge-coloring of K_n with at least $\varphi(n, k)$ edges from each color. If this is not the case, then there will be infinitely many n's for which there is a $\frac{1}{2}$-balanced type-A-coloring of K_n or a type-B-coloring of K_n [**3**] that will not contain a color-consistent copy of G. More precisely, we have the following characterization.

Corollary 3.2 ([**2**]). *Let G be a graph and let r be an integer with $0 < r \leq \lfloor e(G)/2 \rfloor$. Then G is r-tonal if and only if G has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that $e(X, Y), e(G[W]) \in \{r, e(G) - r\}$.*

From this corollary, the following characterizations of balanceable graphs and omnitonal graphs can be derived.

Corollary 3.3 ([**2**])**.** *A graph G is balanceable if and only if G has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that $e(X,Y), e(G[W]) \in \{\lfloor \frac{1}{2}e(G)\rfloor, \lceil \frac{1}{2}e(G)\rceil\}$.*

Corollary 3.4 ([**2**])**.** *A graph G is omnitonal if and only if, for every integer r with $0 \le r \le e(G)$, G has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that $e(X,Y) = e(G[W]) = r$.*

4. Implications

Theorem 3.1 leads to two major questions concerning Ramsey and Turán-type problems. Given a graph G, we are interested in determining both, the minimum number $N = N(r,G)$ and, if the latter exists, the maximum number $\varphi(n,G)$ such that, if $n \ge N$, every 2-edge coloring of K_n with at least $\varphi(n,G)$ edges from each color contains an $(r, e(G)-r)$-copy of G. In [**3, 4, 2**] we concentrated on the determination of certain families of balanceable graphs (i.e., the case $r = e(G)/2$) and in [**2**] also of omnitonal graphs and we have determined the corresponding extremal number and many times, if not the precise number, we have a good estimate of the corresponding Ramsey number, too. The families which were studied include trees, stars, paths and complete graphs.

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