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THE ASYMPTOTICS OF REFLECTABLE WEIGHTED WALKS IN ARBITRARY DIMENSION

M. MISHNA and S. SIMON

Abstract. We consider the weighted lattice walks with a reflectable step set restricted to the positive d-dimensional orthant. We obtain asymptotic formulas for the number of such walks as a function of the weights. To do so, we set up the desired generating function as the diagonal of a rational function. Then we perform a coefficient extraction via an integral computation which is broken up into two cases. One part uses the residue theorem to evaluate the integral within an error, while the other uses known approximations of Fourier-Laplace integrals.

1. Introduction

Lattice walks can model a wide variety of phenomena, yet are simple intuitive objects. We consider combinatorial classes of lattice walks restricted to remain in the positive orthant ($\mathbb{N}^d$). Each model is defined by a set of vectors $\mathcal{S} \subseteq \{-1, 0, 1\}^d$ called a stepset. A walk of length n in the class is a sequence of steps $Z = (Z_1, \ldots, Z_n)$ with $Z_i \in \mathcal{S}$. We view the sequence as incremental moves starting from the origin. Here we focus on stepsets which are *reflectable*, that is, the step set is invariant under reflection across any axis. We require the stepset to define genuinely d-dimensional walks, in the sense that for any dimension there is at least one step that moves in that dimension. The main result (Theorem 2) is a general asymptotic formula which to enumerate weighted walks. The formula is parameterized by the values of the weights.

Reflectable lattice models appear in the literature in the study of walks in Weyl chambers. Several authors have developed formulas for generating functions [**15, 8**] and also enumeration formulas in some cases [**9, 7**]. Our work resembles the analytic approach of Melczer and Mishna [**12**] and Melczer and Wilson [**13**], and the formulas are similarly general, and simply stated. The case where all weights are 1 was considered by Melczer and Mishna, and our formulas agree. The drift of a model is the vector sum of the stepset: $\delta_{\mathcal{S}} := \sum_{\sigma \in \mathcal{S}} \sigma$. By the work of Duraj [**6**], for the walks considered here, when this vector is in the negative orthant $\mathbb{Z}^d_{<0}$, the exponential growth factor and the critical exponent should agree with those found for the excursions of the unweighted model. We show how to prove

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this property in the concluding remarks. The excursion enumeration formulas of Denisov and Wachtel [**5**] agree with ours for the known 2D and 3D cases [**3, 1**]. In two and three dimensions, there are several approaches for asymptotic enumeration of lattice models which pass through differential equations, see [**2**] and the references therein. Differential equation approaches become computationally infeasible in higher dimensions, and present theory does not permit treatment of dimension as a symbolic parameter.

Here, the stepset is weighted using central weights. The weighting could indicate multiple steps in a given direction or probabilities. Courtiel et al. [**4**] showed that the (univariate) ordinary generating function for weighted walks with a central weighting could be obtained as an evaluation of the (multivariate) generating function for unweighted walks considering endpoints. Consequently, we phrase our results in terms of evaluations of the generating function for unweighted walks. We could view this as weighting directions, rather than steps. Other weightings are possible and is work in progress.

1.1. Main result and organization of the extended abstract

We use the following notation. We denote vectors by boldface: $\mathbf{x} := (x_1, \ldots, x_d)$ and extend operations component-wise when it makes sense:

$$\mathbf{x}\boldsymbol{\alpha} := (x_1\alpha_1, \ldots, x_d\alpha_d), \qquad \mathbf{x}^{\boldsymbol{\alpha}} := (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$$
$$\mathbf{x}^{-1} := (x_1^{-1}, \ldots, x_d^{-1}), \qquad e^{\boldsymbol{\theta}} := (e^{\theta_1}, \ldots, e^{\theta_d}).$$

Suppose that $\mathcal{Q}$ is a class of lattice walks. We define the *complete generating function* associated to the model as the formal power series $Q(\mathbf{x};t) := \sum_{\iota\in\mathbb{Z}^d_{\geq 0}, n\geq 0} q(\boldsymbol{\iota};n)\mathbf{x}^{\boldsymbol{\iota}}t^n$, where $q(\boldsymbol{\iota};n)$ is the number of (unweighted) walks of length n that start at the origin end at the point $\boldsymbol{\iota}$.

Proposition 1. *Let $\mathcal{S}$ be any stepset and let $Q(\mathbf{x};t)$ be its associated complete generating function. For any centrally weighted model, there exits a weight-vector $\boldsymbol{\alpha}$ of positive real numbers, and a positive real constant β such that the quantity $q_\alpha(n)$ defined as the weighted sum of all walks of length n is equal to*

$$q_\alpha(n) = [t^n]Q(\boldsymbol{\alpha};\beta t).$$

We define a weighted walk directly using the weight vector $\boldsymbol{\alpha}$ and we assume $\beta = 1$. When $\beta \neq 1$, it suffices to rescale our enumeration results by multiplying the formula by β^n.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ be a vector of positive real numbers. The weight of a walk ending at $\boldsymbol{\iota} \in \mathbb{Z}^d_{\geq 0}$ is the value $\prod \alpha_i^{\iota_i}$. Remark that this is equivalent to weighting a step σ in $\mathcal{S}$ by $\prod_{i=1..d} \alpha_i^{\sigma_i}$ and taking the weight of a walk to be the product of the weights of the steps.

Our main result is the following enumeration formula for weighted reflectable walks in arbitrary dimension.

Theorem 2. *Fix the dimension $d \geq 1$. Let $\mathcal{S} \subset \{-1, 0, 1\}^d$ be a nontrivial reflectable stepset defining a lattice model of walks such that each walk starts at* *the origin and remains in the first orthant* $\mathbb{Z}^d_{\geq 0}$. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ *be a vector of positive weights, and let* $q_\alpha(n) := [t^n]Q(\boldsymbol{\alpha}; t)$ *be the weighted sum of all walks of length* n *as defined above. Asymptotically, as* n *tends to infinity,*

$$q_\alpha(n) \sim \gamma \cdot S(\boldsymbol{\alpha}^+)^n \cdot n^{-(r/2)-m},$$

where $S(x) = \sum_{\boldsymbol{\sigma} \in \mathcal{S}} \mathbf{x}^{\boldsymbol{\sigma}}$, *is the stepset inventory Laurent polynomial;* $\alpha_i^+ = \max\{\alpha_i, 1\}$ *for all* i*;* m *is the number of* α_i *strictly less than 1 and* r *is the number of* α_i *less than or equal to 1, and* γ *is a constant.*

The constant factor of a critical point is the product of the each of the factors $c(\tilde{x}_j)$, given below. In cases where multiple critical points contribute, the constant term γ can depend on the parity of n. Some contributing points have an exponential growth of $(-S(\boldsymbol{\alpha}^+))^n$, so the corresponding constants are added when n is even, and subtracted when n is odd. For a given contributing critical point with component $\tilde{x}_j$ and step set with P_j steps in the positive j direction, the constant term is calculated as:

$$c(\tilde{x}_j) = \begin{cases} 1 - \frac{1}{\alpha_j^2} & \alpha_j > 1 \\ \frac{1}{\sqrt{2 \cdot \pi}} \cdot (2P_j)^{-1/2} \cdot \sqrt{S(\boldsymbol{\alpha}^+)} \cdot 2 & \alpha_j = 1 \\ \frac{1}{\sqrt{2 \cdot \pi}} \cdot (2P_j)^{-3/2} \cdot (S(\boldsymbol{\alpha}^+))^{3/2} \cdot \frac{2}{(1-\alpha_j)^2} & \alpha_j < 1, \tilde{x}_j = \alpha_j \\ \frac{1}{\sqrt{2 \cdot \pi}} \cdot (2P_j)^{-3/2} \cdot (S(\boldsymbol{\alpha}^+))^{3/2} \cdot \frac{2}{(1+\alpha_j)^2} & \alpha_j < 1, \tilde{x}_j = -\alpha_j \end{cases}.$$

The proof uses a description of the generating function as a diagonal of a rational function and applies techniques of analytic combinatorics in several variables. The computation first treats dimensions where the weight is greater than one, and then the weights less than or equal to one. The former are estimated using univariate resides and the latter by an appeal to the substantial theorems.

Example 3 (The simple walks)**.** Consider the three dimensional simple walks, where the step set is the set of elementary vectors, and their negatives:

$$\mathcal{S} = \{\pm(1,0,0), \pm(0,1,0), \pm(0,0,1)\}.$$

The following integer weighting of the steps is central:

Step	$(1,0,0)$	$(-1,0,0)$	$(0,1,0)$	$(0,1,0)$	$(0,0,1)$	$(0,0,1)$
Weight	8	2	4	4	1	16

The associated weight vector is: $\alpha = (2, 1, 1/4)$ with $\beta = 4$, hence $r = 2$, $m = 1$ in Theorem 2. Then as $S(x, y, z) = x + 1/x + y + 1/y + z + 1/z$, we determine that the number of walks of length n has exponential growth $\beta \cdot S(2, 1, 1) = 26$ and subexponential growth $n^{-2/2-1} = n^{-2}$. The associated constant factor is $\frac{169}{6\pi}$.

2. Proof of Theorem 2

Melczer and Mishna outline the strategy of their study of the unweighted case, and the set up here is similar. However, it differs in that we use the two stage evaluation of the integral following the strategy of Courtiel et al. The main steps are as follows:

1. Set up the desired generating function as a diagonal of a rational function;
2. Determine the minimal critical points of the rational function;
3. Write the coefficient as an iterated Cauchy integral;
4. Estimate Cauchy integral by a sum of residues indexed by critical points.

The final step requires potentially intense computations. However, the reflectability of the step set means that the inventory polynomial has a particular form (see Eq. (3)), which significantly simplifies this computation, and allows us to say general things.

The following form of the generating function is developed already in [**12**], and the required to give the weighted version which is simply an evaluation follows Chyzak et al. [**2**]. Here, Δ is the diagonal operator:

$$\Delta \sum_{\mathbf{n}} f(n_1, n_2, \ldots, n_d, n_{d+1}) x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} t^{n_{d+1}} := \sum_{n\geq 0} f(n, n, \ldots, n) t^n$$

which is known to be well defined as applied to these functions, as they are all roughly geometric series.

Proposition 4. *The generating function for weighted walks satisfies:*

$$\begin{aligned} \sum_{n\geq 0} q_\alpha(n) t^n &= \Delta\Big(\frac{G(\mathbf{x})}{H(\mathbf{x},t)}\Big) \\ &= \Delta \frac{\prod_{k=1}^d \alpha_k^{-2}(\alpha_k^2 - x_k^2)}{1 - t(x_1 \cdots x_d) S(\boldsymbol{\alpha}\cdot\mathbf{x}^{-1})} \cdot \frac{1}{(1-x_1)\cdots(1-x_d)}. \end{aligned} \tag{1}$$

We identify $G(\mathbf{x})$ and $H(\mathbf{x}; t)$ as the numerator and denominator of Equation (1) respectively.

The first step is to determine the singular points of $\frac{G(\mathbf{x})}{H(\mathbf{x};t)}$ which contribute to the dominant asymptotic growth. In this case, it is sufficient to find those solutions $\boldsymbol{\rho}^*$ to a particular set of equations, known as *the critical point equations*, which minimize the value $|\rho_1 \ldots \rho_{d+1}|^{-1}$. The first critical point equation is $H(\mathbf{x}; t) = 0$. From this we deduce $t = \frac{1}{x_1 \ldots x_d} S(\boldsymbol{\alpha}\mathbf{x}^{-1})$, since there is only one factor in which t appears. We also see that if $\mathbf{x}^*$ is in the closure of the domain of convergence, each component must satisfy $|x_i^*| \leq 1$ for $1 \leq i \leq d$.

The critical points are also solutions to the following equations:

$$x_1 \frac{\partial H(\mathbf{x}; t)}{\partial x_1} = \cdots = x_d \frac{\partial H(\mathbf{x}; t)}{\partial x_d} = t \frac{\partial H(\mathbf{x}; t)}{\partial t}. \tag{2}$$

The symmetry of the stepset gives $S(\mathbf{x}; t)$ a particular form, which allows us to solve these explicitly. For each k we have:

$$S(\boldsymbol{\alpha}\mathbf{x}^{-1}) = \Big(\frac{\alpha_k}{x_k} + \frac{x_k}{\alpha_k}\Big) P_k(\mathbf{x}) + Q_k(\mathbf{x}) \tag{3}$$

where P_k and Q_k contain *no* x_k. Using this form we see that the equation $x_k \frac{\partial H(\mathbf{x};t)}{\partial x_k} = t \frac{\partial H(\mathbf{x};t)}{\partial t}$ is equivalent to:

$$0 = t x_1 \ldots x_d \cdot \frac{1}{\alpha_k} \cdot (x_k^2 - \alpha_k^2) \cdot P_k(\mathbf{x}). \tag{4}$$

The solution to (4) occurs when either $x_k = \pm\alpha_k$ or $P_k = 0$. The latter possibility is dismissed since it implies that the model has no step in the k-th dimension, contradicting the nontriviality hypothesis.

Proposition 5. *The point* $\mathbf{x}^* = (\boldsymbol{\alpha}^-, t_{\boldsymbol{\alpha}^-})$, *where* $\boldsymbol{\alpha}^- := (\alpha_1^-, \dots, \alpha_d^-)$ *where* $\alpha_k^- = \min\{1, \alpha_k\}$ *and* $t_{\boldsymbol{\alpha}^-} := \frac{1}{\alpha_1^- \dots \alpha_d^- S(\boldsymbol{\alpha}^+)}$ *is a finitely minimal point of* $\frac{G(\mathbf{x})}{H(\mathbf{x};t)}$.

The proof is straightforward analysis of the related critical point equations.

When there are small weights, we must also consider the contribution from points $\tilde{\mathbf{x}}$, identical to $\mathbf{x}^*$ except where components corresponding to small weights are negated. If $|S(\boldsymbol{\alpha}\tilde{\mathbf{x}})| = |S(\boldsymbol{\alpha}\mathbf{x}^*)|$, then we consider its contribution. The results stated above take these additional critical points into account, but for brevity we present how to compute the contribution from the critical point in $\mathbb{R}^d_{>0}$. The computation and analysis for the remaining critical points with negative components is similar.

Critical points contribute to the dominant asymptotics. In general, if $\mathbf{z}^*$ is the unique minimal critical point for $F(\mathbf{z})$, then asymptotically the exponential growth for the coefficients of the diagonal series $\Delta F(\mathbf{z})$ is $|z_1^* \dots z_d^*|$. From Proposition 5, we can conclude that the exponential growth of $q_\alpha(n)$ is $\lim_{n\to\infty} q_\alpha(n)^{1/n} = \left|\alpha_1^- \dots \alpha_d^- \, t_{\boldsymbol{\alpha}^-}\right| = S(\boldsymbol{\alpha}^+)$.

In order to determine the subexponential growth of $q_\alpha(n)$, we express it as an iterated Cauchy integral. We simplify the integral in two stages: first to account for weights greater than 1, and then the weights less than or equal to 1. In order to simplify the presentation, we assume that the weights are in ascending order (reordering the dimensions if necessary). Thus, by the hypotheses, $\alpha_1 \le \dots \le \alpha_m < 1$, $\alpha_{m+1} = \dots = \alpha_r = 1$ and $1 < \alpha_{r+1} \le \dots \le \alpha_d$. We have

$$q_\alpha(n) = [x_1^n][x_2^n]\cdots[x_d^n][t^n]\left(\frac{\prod_{k=1}^d \alpha_k^2(\alpha_k^2 - x_k^2)}{(1 - tx_1 \dots x_d S(\boldsymbol{\alpha}\mathbf{x}^{-1}))\prod_{k=1}^d (1 - x_k)}\right) \tag{5}$$

$$= [x_1^0][x_2^0]\cdots[x_d^0]\left(S(\boldsymbol{\alpha}\mathbf{x}^{-1})^n \prod_{k=1}^d \frac{\alpha_k^2 - x_k^2}{\alpha_k^2\,(1 - x_k)}\right) \tag{6}$$

Then we use the multi-dimensional Cauchy Integral Formula to write this as an integral over a ball centered at the origin, avoiding neighborhoods of the critical points,

$$= \frac{1}{(2\pi\,\mathrm{i})^d}\int \underbrace{S(\boldsymbol{\alpha}\mathbf{x}^{-1})^n \prod_{k=1}^d \frac{\alpha_k^2 - x_k^2}{\alpha_k^2 x_k\,(1 - x_k)}}_{\mathcal{I}(\mathbf{x})}\,\mathrm{d}\,x_d \cdots \mathrm{d}\,x_1.$$

Using the theory of smooth point asymptotics, we apply [**14**, Equation 8.6.2] to express the asymptotics as

$$q_\alpha(n) = \sum_{\tilde{x}} \Phi_{\tilde{x}}(n).$$

The process for determining $\Phi_{\tilde{x}}(n)$ is the same for each critical $\tilde{x}$, so we continue the analysis on the unique positive critical point.

2.1. Large weights

For each dimension in which the weight is more than 1, we can estimate the integral with a residue computation with a controlled error term. In this abstract we will show how to treat the innermost integral, and then repeat this process for all of the dimensions where the weight is greater than 1. This process will result in an expression with r integrals remaining.

In order to estimate the integrals in variables with large weights, we use a residue computation which differs from the original integral by a small enough error term. We sketch how to do this for one variable, x_d, but we can iterate the argument for each variable with large weights. (Or, skip this entirely if $d = r$.)

We can show the integral of $\mathcal{I}(x)$ over $|x_d| = 1 + \epsilon$ has exponential growth strictly less that $S(\boldsymbol{\alpha}^+)$ using some elementary bounds. Therefore, we know that for some constants $K > 0$, and $M_\epsilon < S(\boldsymbol{\alpha}^+)$,

$$\left| \int \cdots \int \int_{|x_d|=1+\epsilon} \mathcal{I}(\mathbf{x}) \,\mathrm{d}\, x_d \cdots \mathrm{d}\, x_1 \right| \leq K\, M_\epsilon^n. \tag{7}$$

Therefore we can subtract off this integral and add an error term of $O(M_\epsilon^n)$, so that we can use the residue theorem inside the region $1 - \epsilon \leq |x_1| \leq 1 + \epsilon$. The only pole is the region is a simple pole is at $x_d = 1$. Thus, the innermost integral evaluates to $2\pi\,\mathrm{i}\,(x_d - 1)\mathcal{I}(\mathbf{x})$ evaluated at $x_d = 1$. Thus,

$$q_\alpha(n) = \frac{(\alpha_d^2 - 1)}{\alpha_d^2 (2\pi\,\mathrm{i})^{d-1}} \cdot \int \cdots \int S\left(\frac{\alpha_1}{x_1}, \ldots, \frac{\alpha_{d-1}}{x_{d-1}}, \alpha_d\right)^n \prod_{k=1}^{d-1} \frac{\alpha_k^2 - x_k^2}{\alpha_k^2 x_k\,(1 - x_k)} \,\mathrm{d}\, x_{d-1} \ldots \mathrm{d}\, x_1 + O(M_\epsilon^n).$$

In short, we see that the dimensions with large weights don't contribute to the subexponential growth.

2.2. Small weights

After processing the large weights we have:

$$q_\alpha(n) = \frac{\prod_{k=r}^{d} (\alpha_d^k - 1)\alpha_k^{-2}}{(2\pi\,\mathrm{i})^r} \cdot \int \cdots \int S\left(\frac{\alpha_1}{x_1}, \ldots, \frac{\alpha_r}{x_r}, \alpha_{r+1}, \ldots, \alpha_d\right)^n \prod_{k=1}^{r} \frac{\alpha_k^2 - x_k^2}{\alpha_k^2 x_k\,(1 - x_k)} \,\mathrm{d}\, x_r \cdots \mathrm{d}\, x_1 + O(M_\epsilon^n).$$

To give exact estimates, we appeal directly to the following theorem of Hörmander [**10**, Theorem 7.7.5], rephrased by Pemantle and Wilson [**14**, Theorem 13.3.2].

Theorem 6 (Hörmander; Pemantle and Wilson)**.** *Suppose that the functions $A(\boldsymbol{\theta})$ and $\phi(\boldsymbol{\theta})$ in d variables are smooth in a neighbourhood $\mathcal{N}$ of the origin and* *that ϕ has a critical point at $\boldsymbol{\theta} = \mathbf{0}$; the Hessian $\mathcal{H}$ of ϕ at $\mathbf{0}$ is non-singular; $\phi(\mathbf{0}) = \mathbf{0}$; and the real part of $\phi(\boldsymbol{\theta})$ is non-negative on $\mathcal{N}$.*

Then for any integer $M > 0$ there are constants $C_0, \cdots C_M$ such that

$$\int_{\mathcal{N}} A(\boldsymbol{\theta})\, \mathrm{e}^{-n\phi(\boldsymbol{\theta})}\, d\boldsymbol{\theta} = \left(\frac{2\pi}{n}\right)^{d/2} \det(\mathcal{H})^{-1/2} \cdot \sum_{j=0}^{M} C_j n^{-j} + O(n^{-M-1}). \tag{8}$$

The constants C_j are given by the formula:

$$C_j = (-1)^j \sum_{\ell \le 2j} \frac{\mathcal{D}^{\ell+j}(A\underline{\phi}^{\ell})(\mathbf{0})}{2^{\ell+j}\ell!(\ell+j)!}, \qquad \textit{with } \underline{\phi} := \phi - \langle \boldsymbol{\theta}, \mathcal{H}\boldsymbol{\theta}\rangle \tag{9}$$

where $\mathcal{D}$ is the differential operator $\mathcal{D} := \langle \nabla, \mathcal{H}^{-1}\nabla\rangle = \sum_{a,b} (\mathcal{H}^{-1})_{a,b} \frac{\partial}{\partial\theta_a}\frac{\partial}{\partial\theta_b}$.

In order to apply this theorem, we first perform a change of variables with the desired critical point centered at the origin. Thus we get a Fourier-Laplace integral with

$$A(\boldsymbol{\theta}) := \prod_{k=1}^{m} \frac{\left(1 - \mathrm{e}^{2\,\mathrm{i}\,\theta_k}\right)}{\left(1 - \alpha_k\, \mathrm{e}^{\mathrm{i}\,\theta_k}\right)} \prod_{k=m+1}^{r} \left(1 + \mathrm{e}^{\mathrm{i}\,\theta_k}\right), \text{ and} \tag{10}$$

(11)

$$\phi(\boldsymbol{\theta}) := \ln\left(S(\boldsymbol{\alpha}^+)\right) - \ln\left(S\left(\frac{\alpha_1}{\mathrm{e}^{\mathrm{i}\,\theta_1}}, \cdots, \frac{\alpha_m}{\mathrm{e}^{\mathrm{i}\,\theta_m}}, \frac{1}{\mathrm{e}^{\mathrm{i}\,\theta_{m+1}}}, \cdots, \frac{1}{\mathrm{e}^{\mathrm{i}\,\theta_r}}, \alpha_{r+1}, \ldots, \alpha_d\right)\right).$$

In order to prove the formula for sub-exponential growth, we must determine the first non-zero value of C_j in the equation above. The following lemma shows the subexponential growth is $n^{-r/2-m}$ as claimed in Theorem 2.

Lemma 6.1. *For weights $\alpha_1, \cdots, \alpha_m < 1$, $\alpha_{m+1} = \cdots = \alpha_r = 1$, and $A, \underline{\phi}$, as defined above, the first j such that C_j in Eq. (9) is nonzero is m, and the only nonzero term in the sum for C_m is $\ell = 0$.*

The proof is mainly computations of derivatives, which is simplified due to the symmetry of the stepset. Combining this lemma with Theorem 6, we calculate that the contribution from a given critical point is

$$\Phi_{\tilde{\mathbf{x}}}(n) \sim \left(\prod_{j=1}^{d} c(\tilde{x}_j)\right) \cdot S(\alpha^+)^n \cdot n^{-(r/2)-m}.$$

3. General observations and future work

Under the same hypotheses as the main theorem, we can give similar formulas for the number of walks in the positive orthant which end on k axes. In particular, the number of excursions in the positive orthant with steps from $\mathcal{S}$ of length n grows as $S(\mathbf{1})^n n^{-3d/2}$. We also note that setting the weights to 1 gives the same asymptotics in the unweighted case given by Theorem 71 of Melczer [**11**].

A similar approach should work to determine general asymptotic formulas for weighted versions of the nearly symmetric walks recently investigated by Melczer and Wilson. More generally, this approach will work for other Weyl groups. This is work in progress. Following [**4**], one can adapt this to consider arbitrary starting points. As in that case, the dominant constant term is then parametrized by the starting point and turns out to be a discrete harmonic function.

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References

1. Bogosel B., Perrollaz V., Raschel K. and Trotignon A., *3D positive lattice walks and spherical triangles*, `arXiv:1804.06245`.
2. Bostan A., Chyzak F., Hoeij M. V., Kauers M. and Pech L., *Hypergeometric expressions for generating functions of walks with small steps in the quarter plane*, European J. Combin. **61** (2017), 242–275.
3. Bostan A., Raschel K. and Salvy B., *Non-D-finite excursions in the quarter plane*, J. Combin. Theory Ser. A **121** (2014), 45–63.
4. Courtiel J., Melczer S., Mishna M. and Raschel K., *Weighted lattice walks and universality classes*, J. Combin. Theory Ser. A **152** (2017), 255–302.
5. Denisov D. and Wachtel V., *Random walks in cones*, Ann. Probab. **43** (2015), 992–1044.
6. Duraj J., *Random walks in cones: the case of nonzero drift*, Stochastic Process. Appl. **124** (2014), 1503–1518.
7. Feierl T., *Asymptotics for the number of zero drift reflectable walks in a Weyl chamber of type A*, `arXiv:1806.05998`.
8. Gessel I. M. and Zeilberger D., *Random walks in a Weyl chamber*, Proc. Amer. Math. Soc. **115** (1992), 27–31.
9. Grabiner D. J., *Random walk in an alcove of an affine Weyl group, and non-colliding random walks on an interval*, J. Combin. Theory Ser. A **97** (2002), 285–306.
10. Hörmander L., *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Springer-Verlag, Berlin, 1990.
11. Melczer S., *Analytic combinatorics in several variables: Effective asymptotics and lattice path enumeration*, `arXiv:1709.05051`
12. Melczer S. and Mishna M., *Asymptotic lattice path enumeration using diagonals*, Algorithmica **75** (2016), 782–811.
13. Melczer S. and Wilson M. C., *Higher dimensional lattice walks: Connecting combinatorial and analytic behavior*, `arXiv:1810.06170`.
14. Pemantle R. and Wilson M. C., *Analytic Combinatorics in Several Variables*, Cambridge University Press, Cambridge, 2013.
15. Zeilberger D., *Andre's reflection proof generalized to the many-candidate ballot problem*, Discrete Math. **44** (1983), 325–326.

M. Mishna, Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada,
e-mail: `mmishna@sfu.ca`

S. Simon, Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada,
e-mail: `ssimon@sfu.ca`