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GROUP OF AUTOMORPHISMS PRESERVING COSETS OF A CENTRAL CHARACTERISTIC SUBGROUP AND RELATED RESULTS

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Abstract. A concept of Subcentral automorphisms of a group G with respect to a characteristic subgroup M of $Z(G)$ along with relevant mathematical paraphernalia is introduced. With the help of this a number of results on central automorphisms have been generalized.

1. Introduction

Let G be a group. We shall denote the commutator, centre, group of automorphisms and group of inner automorphisms of G by G', $Z(G)$, $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$, respectively. Let $\exp(G)$ denote the exponent of G.

For any group H and abelian group K, let $\mathrm{Hom}(H, K)$ denote the group of all homomorphisms from H to K. This is an abelian group with binary operation $fg(x) = f(x)g(x)$ for $f, g \in \mathrm{Hom}(H, K)$.

An automorphism α of G is called central if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$. The set of all central automorphisms of G, which is here denoted by $\mathrm{Aut}_Z(G)$, is a normal subgroup of $\mathrm{Aut}(G)$. Notice that $\mathrm{Aut}_Z(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$, the centralizer of the subgroup $\mathrm{Inn}(G)$ in the group $\mathrm{Aut}(G)$. The elements of $\mathrm{Aut}_Z(G)$ act trivially on G'.

There has been a number of results on the central automorphisms of a group. M. J. Curran [**2**] proved: *for any non-abelian finite group G, $\mathrm{Aut}_Z^Z(G)$ is isomorphic with $\mathrm{Hom}(G/G'Z(G), Z(G))$, where $\mathrm{Aut}_Z^Z(G)$ is group of all those central automorphisms which preserve the center $Z(G)$ elementwise.*

In [**3**], Franciosi et al. showed that: *if $Z(G)$ is torsion-free and $Z(G)/G' \cap Z(G)$ is torsion, then $\mathrm{Aut}_Z(G)$ acts trivially on $Z(G)$. It is an abelian and torsion-free group.* They further proved that *$\mathrm{Aut}_Z(G)$ is trivial when $Z(G)$ is torsion-free and G/G' is torsion.* In [**5**], Jamali et al. proved that *for a finite group G in which $Z(G) \leq G'$, $\mathrm{Aut}_Z(G) \cong \mathrm{Hom}(G/G', Z(G))$.* They also proved that *if G is a purely non-abelian finite p-group of class two (p odd), then $\mathrm{Aut}_Z(G)$ is elementary abelian if and only if $\Omega_1(Z(G)) = \phi(G)$, and $\exp(Z(G)) = p$ or $\exp(G/G') = p$,*

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where $\phi(G)$ is Frattini subgroup of G and $\Omega_1(Z(G)) = \langle x \in Z(G) | x^p = 1\rangle$. Note that a group G is called purely non-abelian if it has no non trivial abelian direct factor. Adney [**1**] proved: *if a finite group G has no abelian direct factor, then there is a one-one and onto map between* $\mathrm{Aut}_Z(G)$ *and* $\mathrm{Hom}(G, Z(G))$.

In this article, we generalize the above results to subcentral automorphisms.

2. Subcentral Automorphisms

Let M and N be two normal subgroups of G. By $\mathrm{Aut}_N(G)$, we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which induce identity on G/N. By $\mathrm{Aut}^M(G)$, we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which restrict to the identity on M.

Let $\mathrm{Aut}_N^M(G) = \mathrm{Aut}_N(G) \cap \mathrm{Aut}^M(G)$. From now onwards, M will be a characteristic central subgroup and elements of $\mathrm{Aut}_M(G)$ will be called subcentral automorphisms of G (with respect to subcentral subgroup M). It can be seen that $\mathrm{Aut}_M(G)$ is a normal subgroup of $\mathrm{Aut}_Z(G)$.

We further let

$$C^* = \{\alpha \in \mathrm{Aut}^M(G) : \alpha\beta = \beta\alpha, \quad \text{for all } \beta \in \mathrm{Aut}_M(G)\}.$$

Clearly C^* is a normal subgroup of $\mathrm{Aut}(G)$. Since every inner automorphism commutes with elements of $\mathrm{Aut}_Z(G)$, $\mathrm{Inn}(G) \leq C^*$. Let

$$P = \langle \{[g, \alpha] : g \in G, \alpha \in C^*\}\rangle, \qquad \text{where } [g, \alpha] \equiv g^{-1}\alpha(g).$$

In the following, P and C^* will always correspond to a central subgroup of M of G as in the above definitions.

Following proposition shows that each element of P is invariant under the natural action of $\mathrm{Aut}_M(G)$.

Proposition 2.1. $\mathrm{Aut}_M(G)$ *acts trivially on* P.

Proof. Consider an automorphism $\alpha \in \mathrm{Aut}_M(G)$. This implies $x^{-1}\alpha(x) \in M$ for all $x \in G$. So $\alpha(x) = xm$ for some $m \in M$. Let $\beta \in C^*$. By definition of C^*, we have

$$\begin{aligned}\alpha([x, \beta]) = \alpha(x^{-1}\beta(x)) &= (\alpha(x))^{-1}\beta(\alpha(x)) \\ &= m^{-1}x^{-1}\beta(xm) \quad (\because \alpha \in \mathrm{Aut}_M(G)) \\ &= m^{-1}x^{-1}\beta(x)m = x^{-1}\beta(x) = [x, \beta], \quad (\because \beta \in C^*).\end{aligned}$$

Hence the result follows. □

Let E^* be any normal subgroup of $\mathrm{Aut}(G)$ contained in C^*, and $K = \langle [g, \alpha] | g \in G, \alpha \in E^*\rangle$.

In particular, when $E^* = \mathrm{Inn}(G)$, we get $K = G'$. In the following, K represents a subgroup of P which is obtained in the above manner for a corresponding E^*. Since K is a subgroup of P, it is invariant under the action of $\mathrm{Aut}_M(G)$. It is easy to see that K is a characteristic subgroup of G, and hence it is a normal subgroup of G.

Our main results are given by the following theorems.

Theorem A. *For a finite group G,* $\mathrm{Aut}_M^M(G) \cong \mathrm{Hom}\,(G/KM, M)$.

Theorem B. *Let G be a group with M torsion-free and $M/M \cap K$ torsion. Then* $\mathrm{Aut}_M(G)$ *is a torsion-free abelian group which acts trivially on M.*

Theorem C. *If G is a purely non-abelian finite group, then* $|\mathrm{Aut}_M(G)| = |\mathrm{Hom}(G, M)|$.

Theorem D. *If G is a purely non-abelian finite p-group (p odd), then* $\mathrm{Aut}_M(G)$ *is an elementary abelian p-group if and only if* $\exp(M) = p$ *or* $\exp(G/K) = p$.

Theorem E. *Let G be a non-abelian finitely generated group in which M is indecomposable with both M and G/M torsion free abelian, then* $\mathrm{Hom}(G, M)$ *is a torsion free abelian group.*

Proof of Theorem A. For any $\mu \in \mathrm{Aut}_M^M(G)$, define the map

$$\psi_\mu \colon \mathrm{Hom}\left(\frac{G}{KM}, M\right) \to M \qquad \text{as} \qquad \psi_\mu(gKM) = g^{-1}\mu(g).$$

We first show that ψ_μ is well defined. Let $gKM = hKM$, i.e., $gh^{-1} \in KM$. Therefore,

$$\mu(gh^{-1}) = gh^{-1} \quad \Longrightarrow \quad g^{-1}\mu(g) = h^{-1}\mu(h) \quad \Longrightarrow \quad \psi_\mu(gKM) = \psi_\mu(hKM).$$

For proving ψ_μ is a homomorphism, consider

$$\begin{aligned}\psi_\mu(gKMhKM) &= \psi_\mu(ghKM) = (gh)^{-1}\mu(gh) = h^{-1}g^{-1}\mu(g)\mu(h)\\ &= g^{-1}\mu(g)h^{-1}\mu(h) = \psi_\mu(gKM) \cdot \psi_\mu(hKM).\end{aligned}$$

Now define a map $\psi \colon \mathrm{Aut}_M^M(G) \to \mathrm{Hom}\left(\frac{G}{KM}, M\right)$ as $\psi(\mu) = \psi_\mu$. We show that ψ is the required isomorphism. For $f, g \in \mathrm{Aut}_M^M(G)$ and $h \in G$,

$$\begin{aligned}\psi(fg)(hKM) &= \psi_{fg}(hKM) = h^{-1}fg(h) = h^{-1}f(hh^{-1}g(h))\\ &= h^{-1}f(h)h^{-1}g(h) = \psi_f(hKM)\psi_g(hKM) = \psi_f \cdot \psi_g(hKM).\end{aligned}$$

Hence $\psi(fg) = \psi(f)\psi(g)$.

Consider $\psi(\mu_1) = \psi(\mu_2)$, i.e, $\psi_{\mu_1}(gKM) = \psi_{\mu_2}(gKM)$ for all $g \in G$. This implies $g^{-1}\mu_1(g) = g^{-1}\mu_2(g)$, so $\mu_1 = \mu_2$ as g is an arbitrary element of G. Thus ψ is a monomorphism.

We next show that ψ is onto. For any $\tau \in \mathrm{Hom}\left(\frac{G}{KM}, M\right)$, define a map $\mu \colon G \to G$ as $\mu(g) = g\tau(gKM)$ and $g \in G$. Now we show that $\mu \in \mathrm{Aut}_M^M(G)$. For $g_1, g_2 \in G$,

$$\mu(g_1g_2) = g_1g_2\tau(g_1g_2KM) = g_1\tau(g_1KM)g_2\tau(g_2KM) = \mu(g_1)\mu(g_2).$$

Therefore, μ is a homomorphism on G.

Further, let $\mu(g) = 1$. This implies

$$g\tau(gKM) = 1 \quad \Longrightarrow \quad \tau(gKM) = g^{-1} \quad \Longrightarrow \quad g^{-1} \in M.$$

Therefore,

$$gKM = KM \quad \Longrightarrow \quad \tau(gKM) = 1 \quad \Longrightarrow \quad g = 1.$$

Hence μ is one-one.

As G is finite, μ must be onto. So $\mu \in \mathrm{Aut}(G)$. Further, as $g^{-1}\mu(g) = g^{-1}g\tau(gKM) = \tau(gKM) \in M$, so $\mu \in \mathrm{Aut}_M(G)$. Also if $g \in M$, then

$$\mu(g) = g(\tau(gKM)) = g\tau(KM) = g.$$

Thus, $\mu \in \mathrm{Aut}_M^M(G)$ and $\psi(\mu) = \tau$. □

Hence the theorem follows.

Corollary 2.2. *Let G be a finite group with $M \le K$, then* $\mathrm{Aut}_M(G) \cong \mathrm{Hom}(G/K, M)$.

Proof. Since $M \le K$ $G/KM = G/K$. The result follows directly from Theorem A and Proposition 2.1. □

Proof of Theorem B. Let $\alpha \in \mathrm{Aut}_M(G)$. If x is an element of M, then by the hypothesis, $x^n \in M \cap K$ for some positive integer n. By Proposition 2.1, we have $x^n = \alpha(x^n) = (\alpha(x))^n$, and hence $x^{-n}(\alpha(x))^n = 1$. Since $x^{-1}\alpha(x) \in M \subseteq Z(G)$, this implies $(x^{-1}\alpha(x))^n = 1$. As M is torsion-free, this implies that $x^{-1}\alpha(x) = 1$, i.e., $\alpha(x) = x$. Therefore, $\mathrm{Aut}_M(G)$ acts trivially on M.

Let $\alpha, \beta \in \mathrm{Aut}_M(G)$ and $x \in G$. So

$$\begin{aligned}\alpha\beta(x) &= \alpha(\beta(x)) = \alpha(xx^{-1}\beta(x)) = \alpha(x)x^{-1}\beta(x) = xx^{-1}\alpha(x)x^{-1}\beta(x)\\ &= \beta(x)x^{-1}\alpha(x) = \beta(x)\beta(x^{-1}\alpha(x)) = \beta\alpha(x).\end{aligned}$$

Thus, $\mathrm{Aut}_M(G)$ is an abelian group.

Now, consider $\alpha \in \mathrm{Aut}_M(G)$ and suppose there exists $k \in N$ such that $\alpha^k = 1$. Since $x^{-1}\alpha(x) \in M$ for all $x \in G$, there exists $g \in M$ such that $\alpha(x) = xg$. Further, $\alpha^2(x) = \alpha(\alpha(x)) = \alpha(xg) = \alpha(x)\alpha(g) = xg^2$ (because α acts trivially on M). Hence, by induction, $\alpha^n(x) = xg^n$. But $\alpha^k = 1$ implies $x = xg^k$, i.e, $g^k = 1$. As M is torsion-free, we must have $g = 1$. Thus $\alpha(x) = x$ for every x, i.e., $\alpha = 1$. Therefore, $\mathrm{Aut}_M(G)$ is torsion-free, and the theorem follows. □

Proposition 2.3. *If G is a group in which M is torsion-free and G/K is torsion, then* $\mathrm{Aut}_M(G) = 1$.

Proof. Let $\alpha \in \mathrm{Aut}_M(G)$ and $x \in G$. Then by the assumption, $x^n \in K$ for some $n \in N$. As α fixes K elementwise, we have $(\alpha(x))^n = \alpha(x^n) = x^n$. So $x^{-n}(\alpha(x))^n = 1$. But $\alpha \in \mathrm{Aut}_M(G)$, and hence $x^{-1}\alpha(x) \in M \le Z(G)$. This implies that $(x^{-1}\alpha(x))^n = 1$. Since M torsion-free, it follows that $x^{-1}\alpha(x) = 1$, i.e., $\alpha(x) = x$, for all $x \in G$. So $\mathrm{Aut}_M(G) = 1$. □

Proof of Theorem C. For $f \in \mathrm{Aut}_M(G)$, we let $\alpha(f) \equiv \alpha_f$ defined as $\alpha(f)(g) \equiv \alpha_f(g) = g^{-1}f(g), g \in G$. It can be shown that $\alpha_f \in \mathrm{Hom}(G, M)$. We thus have $\alpha\colon \mathrm{Aut}_M(G) \to \mathrm{Hom}(G, M)$. One can easily see that α is injective.

It just remains to show that α is onto. For $\sigma \in \mathrm{Hom}(G, M)$, consider the map $f\colon G \to G$ given by $f(g) = g\sigma(g)$. f is an endomorphism and also $g^{-1}f(g) = \sigma(g) \in M$, which implies that f is a subcentral endomorphism of G, and hence f

is a normal endomorphism, (i.e., f commutes with all inner automorphisms). So, clearly $\mathrm{Im}(f)$ is a normal subgroup of G.

It is easy to see that f^n is also normal endomorphism, and hence $Im(f^n)$ is a normal subgroup of G for all $n \geq 1$. Since G is a finite group, the two series

$$\mathrm{Ker}\, f \leq \mathrm{Ker}\, f^2 \leq \dots$$

$$\mathrm{Im}(f) \geq \mathrm{Im}(f^2) \geq \dots$$

will terminate.

So there exists $k \in N$ such that

$$\mathrm{Ker}\, f^k = \mathrm{Ker}\, f^{k+1} = \dots = A,$$

$$\mathrm{Im}(f^k) = \mathrm{Im}(f^{k+1}) = \dots = B.$$

Now, we prove that $G = AB$.

Let $g \in G$, $f^k(g) \in \mathrm{Im}(f^k) = \mathrm{Im}(f^{2k})$, and so $f^k(g) = f^{2k}(h)$ for some $h \in G$. Therefore, $f^k(g) = f^k(f^k(h))$. This implies $f^k(g^{-1})f^k(g) = f^k(g^{-1})f^k(f^k(h))$. Thus $(f^k(h))^{-1}g \in \mathrm{Ker}\, f^k = A$. Thus $g \in AB$, and hence $G = AB$.

Clearly $A \cap B = \langle 1 \rangle$ and therefore, $G = A \times B$. If $f(g) = 1$, then $g^{-1}\sigma(g) = 1$. This implies $\mathrm{Ker}\, f \leq M$. Similarly, if $f^2(g) = 1$, i.e., $f(f(g)) = 1$. Thus $f(g) \in \mathrm{Ker}\, f \leq M$. Therefore, $g\sigma(g) \in M$ implies $g \in M$. Hence $\mathrm{Ker}\, f^2 \leq M$. Repetition of this argument gives $A \equiv \mathrm{Ker}\, f^k \leq M \leq Z(G)$. This implies A is an abelian group. By assumption, G is purely non-abelian, and hence we must have $A \equiv \mathrm{Ker}\, f^k = 1$. This further implies $\mathrm{Ker}\, f = 1$, i.e., f is injective. So $G = B \equiv \mathrm{Im}(f^k) = \mathrm{Im}(f)$. Thus f surjective. Hence, $f \in \mathrm{Aut}_M(G)$. From the definition of α, it follows that $\alpha(f) = \sigma$. Hence α is surjective. Therefore, α is the required bijection. Hence the result follows. □

Proposition 2.4. *Let G be a purely non-abelian finite group, then for each $\alpha \in \mathrm{Hom}(G, M)$ and each $x \in K$, we have $\alpha(x) = 1$. Further $\mathrm{Hom}(G/K, M) \cong \mathrm{Hom}(G, M)$.*

Proof. Whenever G is a purely non-abelian group, then by Theorem C, $|\mathrm{Aut}_M(G)| = |\mathrm{Hom}(G, M)|$. For every $\sigma \in \mathrm{Aut}_M(G)$, it follows that $f_\sigma\colon x \to x^{-1}\sigma(x)$ is a homomorphism from G to M. Further the map $\sigma \to f_\sigma$ is one-one, and thus a bijection because $|\mathrm{Aut}_M(G)| = |\mathrm{Hom}(G, M)|$. So every homomorphism from G to M can be considered as an image of some element of $\mathrm{Aut}_M(G)$ under this bijection. Let $\alpha \in \mathrm{Hom}(G, M)$. Since $K = \{[g, \alpha] : g \in G,\ \alpha \in C^*\}$, a typical generator of K is given by $g^{-1}\beta(g)$ for an element $g \in G$, and $\beta \in C^*$. So $\alpha(g^{-1}\beta(g)) = f_\sigma(g^{-1}\beta(g)) = (g^{-1}\beta(g))^{-1}\sigma(g^{-1}\beta(g))$. But by Proposition 2.1, $\sigma(g^{-1}\beta(g)) = g^{-1}\beta(g)$ (because $g^{-1}\beta(g) \in K$), and hence $\alpha(g^{-1}\beta(g)) = \beta^{-1}(g)gg^{-1}\beta(g) = 1$. It follows that $\alpha(x) = 1$ for every $x \in K$.

Now consider the map $\phi\colon \mathrm{Hom}(G, M) \to \mathrm{Hom}(G/K, M)$ such that $\phi(f) = \bar{f}$, where $\bar{f}(gK) = f(g)$ for all $g \in G$. Clearly this map ϕ is an isomorphism. □

Proposition 2.5. *Let G be a purely non-abelian finite group, then $|\mathrm{Aut}_M(G)| = |\mathrm{Hom}(G/K, M)|$.*

Proof. Proof follows directly from Theorem C and Proposition 2.4. □

Proposition 2.6. *Let p be a prime number. If G is a purely non-abelian finite p-group, then $\mathrm{Aut}_M(G)$ is a p-group.*

Proof. By the assumption, the subgroup M, and hence $\mathrm{Hom}(G/M, M)$ are finite p-groups. Hence the result follows directly from Proposition 2.5. □

Proposition 2.7. *Let G be a purely non-abelian finite group.*
(i) *If* $\gcd(|G/K|, |M|) = 1$*, then* $\mathrm{Aut}^M(G) = 1$.
(ii) *If* $\mathrm{Aut}_M(G) = 1$*, then* $M \leq K$.

Proof. (i) Follows from Proposition 2.5.
(ii) Let $|G/K| = a$ and $|M| = b$. Since $\mathrm{Aut}_M(G) = 1$, hence by Proposition 2.5, $(a, b) = 1$. So there exist integers λ and μ such that $\lambda a + \mu b = 1$. Let $x \in M$. Thus $xK = (xK)^1 = (xK)^{\lambda a+\mu b} = (xK)^{\lambda a}(xK)^{\mu b} = K$, so $x \in K$. □

Remark 2.1. From Corollary 2.2 and Proposition 2.4, we can say that whenever $M \leq K$, $\mathrm{Aut}_M(G) \cong \mathrm{Hom}(G, M)$. Even when $\mathrm{Im}\, f \leq K$, for all $f \in \mathrm{Hom}(G, M)$, this result holds. Thus, if G is a purely non-abelian finite group and if for all $f \in \mathrm{Hom}(G, M)$, $\mathrm{Im}\, f \leq K$, then $\mathrm{Aut}_M(G) \cong \mathrm{Hom}(G/K, M)$.

Remark 2.2. For every $f \in \mathrm{Hom}(G, M)$, the map $\sigma_f : x \to xf(x)$ is a subcentral endomorphism of G. This endomorphism is an automorphism if and only if $f(x) \neq x^{-1}$ for all $1 \neq x \in G$ (G is finite).

Adney and Yen [**1**] proved that the mapping

$$f(x) = x^{-1} \text{ for all } 1 \neq x \in G, \qquad \text{and} \qquad f \in \mathrm{Hom}(G, Z(G))$$

does not exist for a purely non-abelian finite p-group. Hence, for a purely non-abelian p-group, σ_f is an automorphism.

The following lemma was proved in [**4**]. We use it to prove Theorem D.

Lemma 2.8. *Let x be an element of a finite p-group G and N a normal subgroup of G containing G' such that $o(x) = o(xN) = p$. If the cyclic subgroup $< x >$ is normal in G such that $ht(xN) = 1$, then $\langle x\rangle$ is a direct factor of G.*

In the above statement, ht denotes height. Height of an element x of a p-group G is defined as the largest p-power p^n such that $x \in G^{p^n}$, where $G^m = \{g^m \ : \ g \in G\}$.

Proof of Theorem D. For an odd prime p, let $\mathrm{Aut}_M(G)$ be an elementary abelian p-group. Assume that the exponents of M and G/K are both strictly greater than p. Since G/K is finite abelian, it has a cyclic direct summand $\langle xK\rangle$, say, of order $p^n (n \geq 2)$, and hence there exists L so that $G/K \cong \langle xK\rangle \times L/K$. For $f \in \mathrm{Hom}(G, M)$, there exists an element a of order p^m, where $2 \leq m \leq n$ such that $f(x) = a$ for any $x \in G$ (From Remark 2.1, we can easily see that $a \neq x^{-1}$). So $\bar{f}(xK) = a$.

We can use the homomorphism $\bar{f}$ to get a corresponding homomorphism (also denoted by the same notation) $\bar{f}$ as $\bar{f} : \langle xK\rangle \times L/K \to M$ with $(x^i K, lK) \to a^i$. The map $\bar{f}$ on $\langle xK\rangle \times L/K$ is well defined since $o(a)|o(xK)$ (as $m \leq n$). If $aK = (x^s K, lK)$, then we show that $p|s$. Assume $p \nmid s$, then $\langle xK\rangle = \langle x^s K\rangle$, and hence $G/K = \langle aK\rangle L/K$. Now we have $o(a) \geq o(aK) \geq o(x^sK) = o(xK) \geq o(\bar{f}(xK)) = o(a)$. This implies that $o(a) = o(aK)$. Thus $\langle a\rangle \cap K = 1$. As $o(aK) = o(xK)$, we get $G/K \cong \langle aK\rangle \times L/K$, and hence $G \cong \langle a\rangle \times L$. This is a contradiction as G is a purely non-abelian group. Thus $p|s$.

By Remark 2.2 and Theorem C, $\sigma_f \in \mathrm{Aut}_M(G)$ and by assumption, $o(\sigma_f) = p$. Now, we have $\sigma_f(x) = xf(x) = xa$. Since $f(a) = \bar{f}((xK)^s, lK) = a^s$, we have

$$\sigma_f^2(x) = xa^{s+2} = xa^{\frac{(s+1)^2-1}{s}} = xa^{n_2(s)},$$

where $n_j(s) = \frac{(s+1)^j-1}{s}$ for $j \in \mathbb{N}$. Also, $\sigma_f^3(x) = xa^{n_3(s)}$.

Generalizing this, we get $\sigma_f^t(x) = xa^{n_t(s)}$ for every $t \in \mathbb{N}$.

As the order of σ_f is p, we have $a^{n_p(s)} = 1$. Since p is odd and $p|s$, we have $p^2|(n_p(s) - p)$. Therefore, $qp^2 + p = n_p(s)$ for some $q \in \mathbb{Z}$. Thus $(a^p)^{qp+1} = 1$. But $o(a) = p^m$ implies $o(a^p) = p^{m-1}$. Now

(1) if $a^p \neq 1$, then $p^{m-1}|(qp + 1)$. But this is impossible as $m \geq 2$.

(2) $a^p = 1$ is also not possible as $o(a) = p^m$ and $m \geq 2$.

So, the assumption that $\exp(M)$ and $\exp(G/M)$ are stricly greater than p is wrong.

Conversely, assume that $\exp(G/K) = p$ and $f \in \mathrm{Hom}(G, M)$. Then by proposition, $\bar{f} \in \mathrm{Hom}(G/K, M)$. So for $x \in G$, put $\bar{f}(xK) = a$. If $aK \neq 1$, then it follows that $o(aK) = o(a) = p$. Clearly $\langle a\rangle \leq M \leq Z(G)$, and hence the cyclic subgroup $\langle a\rangle$ is normal in G. We also have $ht(aK) = 1$. Now by the Lemma 2.8, the cyclic subgroup $\langle a\rangle$ is an abelian direct factor of G, and this contradicts the assumption. Therefore, $a \in K$. This implies that $\mathrm{Im}(f) \leq K$. Hence by Remark 2.1, $\mathrm{Aut}_M(G) \cong \mathrm{Hom}(G/K, M)$. But as M is abelian, $\mathrm{Hom}(G/K, M)$ is abelian. Thus $\mathrm{Aut}_M(G)$ is abelian. Since $\exp(G/K) = p$, this implies that $\mathrm{Aut}_M(G)$ is an elementary abelian p-group.

Now assume that $\exp(M) = p$. Consider $f, g \in \mathrm{Hom}(G, M)$. We first show that $g \circ f(x) = 1$ for all $x \in G$. Assume that $\bar{f}(xK) = b \in M$ for $x \in G$. Since $\exp(M) = p$, it implies that $o(b)|p$. If $b = 1$, then $g \circ f(x) = g(\bar{f}(xK)) = 1$. Now take $o(b) = p$. If $b \in K$, then we have $g(f(x)) = g(\bar{f}(xK)) = g(b) = 1$. Assume b does not belong to K. As $b^p = 1$, it follows that $o(bK) = p$. Also, as $b \in M \leq Z(G)$, $\langle b\rangle$ is normal in G. Now if $ht(bK) = 1$, then by the Lemma 2.8, the cyclic subgroup $\langle b\rangle$ is an abelian direct factor of G, giving a contradiction. So assume $ht(bk(G)) = p^m$ for some $m \in \mathbb{N}$. By the definition of height, there exists an element yK in G/K such that $bK = (yK)^{p^m}$. But $\exp(M) = p$. Therefore, $g \circ f(x) = g(b) = \bar{g}(bK) = \bar{g}((yK)^{p^m}) = 1$. Thus, for all $f, g \in \mathrm{Hom}(G, M)$ and each $x \in G$, $g(f(x)) = 1$. We can similarly show that $f(g(x)) = 1$, and hence $f \circ g = g \circ f$. From Remark 2.2, $\sigma_f \circ \sigma_g = \sigma_g \circ \sigma_f$. This shows that $\mathrm{Aut}_M(G)$ is abelian.

Now we show that each non trivial element of $\mathrm{Aut}_M(G)$ has order p. So if $\alpha \in \mathrm{Aut}_M(G)$, then by Remark 2.2, there exists a homomorphism $f \in \mathrm{Hom}(G, M)$ such that $\alpha = \sigma_f$. Therefore, we have to show that $o(\sigma_f)|p$. Clearly, taking $f = g$ and using $f(f(x)) = 1, x \in G$, we have $x \in G$ and $\sigma_f^2(x) = \sigma_f(xf(x)) = x(f(x))^2$.

In general, for $n \geq 1$, $\sigma_f^n(x) = x(f(x))^n$. As $\exp(M) = p$ and $f(x) \in M$, we have $\sigma_f^p(x) = x$ which implies $\sigma_f^p = 1_{\mathrm{Aut}_M(G)}$.

Hence $o(\sigma_f)|p$. Thus, $o(\alpha)|p$ for all $\alpha \in \mathrm{Aut}_M(G)$. Therefore, $\mathrm{Aut}_M(G)$ is an elementary abelian group. □

Lemma 2.9. *Let G be a non-abelian finitely generated group such that G/M and M are torsion free abelian. Suppose that M is indecomposable and $f \in \mathrm{Hom}(G, M)$, then $M \leq \mathrm{Ker}(f)$.*

Proof. Since G/M and M are abelian, $G' \leq M \cap Ker(f)$. Therefore, $G/M \cap \mathrm{Ker}(f)$ is abelian.

The map $\sigma\colon x(M \cap \mathrm{Ker}(f)) \to xM$ defines an epimorphism from $G/M \cap \mathrm{Ker}(f)$ onto G/M.

Since G/M is a free abelian group, by [**6**, Theorem 4.2.4] there exists a homomorphism $\alpha\colon G/M \to G/M \cap \mathrm{Ker}(f)$ such that $\sigma \circ \alpha$ is an identity on G/M.

Since $\mathrm{Im}(\alpha)$ is a subgroup of $G/M \cap \mathrm{Ker}(f)$, there exists a subgroup L of G containing $M \cap \mathrm{Ker}(f)$ such that $\mathrm{Im}(\alpha) = L/M \cap \mathrm{Ker}(f)$. Because $G/M \cap \mathrm{Ker}(f)$ is abelian, $L/M \cap \mathrm{Ker}(f)$ is a normal subgroup of $G/M \cap \mathrm{Ker}(f)$. So that L is a normal subgroup of G.

Here $G = ML$. This follows from following arguments.
Here α is an injective homomorphism from G/M to $G/M \cap \mathrm{Ker}(f)$ and its image is $L/M \cap \mathrm{Ker}(f)$. This means, if we pull back L via inverse of α, then we get G. But the inverse image is ML. Hence $G = ML$.

Also, $M \cap L = M \cap \mathrm{Ker}(f)$ because $\sigma \circ \alpha$ is an identity on G/M. Here $L \neq 1$, otherwise $G = M$, and so G is abelian, which is a contradiction as G is non abelian.

From the fact that G is finitely generated, it follows that $L/M \cap \mathrm{Ker}\ f$, and so $M/M \cap \mathrm{Ker}(f)$ is a finitely generated abelian group.

Furthermore, $M/M \cap \mathrm{Ker}(f)$ is torsion free. Let $t \in M$ and $k \in N$ be such that

$$(t(M \cap \mathrm{Ker}(f)))^k = 1.$$

Since M is torsion free, $f(t) = 1$, and so $t \in \mathrm{Ker}(f)$. Therefore, $t \in M \cap \mathrm{Ker}(f)$. This shows that $M/M \cap \mathrm{Ker}(f)$ is free abelian. Hence by [**6**, Theorem 4.2.5], $M = (M \cap \mathrm{Ker}(f)) \times A$ for some $A \leq M$. Since M is indecomposable, $M \cap \mathrm{Ker}\ (f) = 1$ or $A = 1$.

If $M \cap \mathrm{Ker}(f) = 1$, then $G' = 1$, and so G is abelian. This is a contradiction.

Therefore, we must have $A = 1$, and this means $M = M \cap \mathrm{Ker}(f)$. Thus, M is contained in $\mathrm{Ker}(f)$ as desired. □

Proof of Theorem E. Let $f \in \mathrm{Hom}(G, M)$. Consider a map $\sigma_f\colon G/M \to M$ as $\sigma_f(x) = f(x)$. It defines a homomorphism from G/M to M.

Hence σ_f is well defined. This is because, for $x_1, x_2 \in G$, $x_1 x_2^{-1} \in M$ implies $f(x_1 x_2^{-1}) = 1$ by Lemma 2.9, and this means $f(x_1) = f(x_2)$. Clearly, σ_f is a homomorphism. Thus, $\sigma_f \in \mathrm{Hom}(G/M, M)$.

Now, it is easy to see that the map $f \to \sigma_f$ is an isomorphism from $\mathrm{Hom}(G, M)$ to $\mathrm{Hom}(G/M, M)$. Therefore,

$$\mathrm{Hom}(G, M) \cong \mathrm{Hom}(G/M, M).$$

Since G/M is a free abelian group, there exists $n \in N$ such that $G/M = \underbrace{Z \times Z \times \ldots \times Z}_{n \text{ times}}$.

Therefore,

$$\begin{aligned} \operatorname{Hom}(G, M) &\backsimeq \operatorname{Hom}(G/M, M) \\ &\backsimeq \operatorname{Hom}(\underbrace{Z \times Z \times \ldots \times Z}_{n \text{ times}}, M) \\ &\backsimeq \operatorname{Hom}\underbrace{(Z, M) \times \ldots \times \operatorname{Hom}(Z, M)}_{n \text{ times}} \\ &\backsimeq \underbrace{M \times M \times \ldots \times M}_{n \text{ times}} \end{aligned}$$

by [**6**, Theorem 4.7 and 4.9].

Since M is a torsion-free abelian group, $\operatorname{Hom}(G, M)$ is a torsion free abelian group. □

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