THE SIZE-RAMSEY NUMBER
OF POWERS OF BOUNDED DEGREE TREES

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Abstract. Given an integer \( s \geq 1 \), the \( s \)-colour size-Ramsey number
of a graph \( H \) is the smallest integer \( m \) such that there exists a graph \( G \) with \( m \) edges with the
property that, in any colouring of \( E(G) \) with \( s \) colours, there is a monochromatic
copy of \( H \). We prove that, for any positive integers \( k \) and \( s \), the \( s \)-colour size-Ramsey
number of the \( k \)th power of any \( n \)-vertex bounded degree tree is linear in \( n \).

1. Introduction

Given graphs \( G \) and \( H \), and a positive integer \( s \), we denote by \( G \to (H)_s \) the
property that any \( s \)-colouring of the edges of \( G \) contains a monochromatic copy of \( H \). We are interested in the problem proposed by Erdős, Faudree, Rousseau and
Schelp [9] of determining the minimum integer \( m \) for which there is a graph \( G \) with \( m \) edges such that property \( G \to (H)_2 \) holds. Formally, the \( s \)-colour size-Ramsey number \( \hat{r}_s(H) \) of a graph \( H \) is defined as follows:

\[
\hat{r}_s(H) := \min\{|E(G)| : G \to (H)_s\}.
\]

Answering a question posed by Erdős [8], Beck [2] showed that \( \hat{r}_2(P_n) = O(n) \)
by means of a probabilistic proof. Alon and Chung [1] proved the same fact by
explicitly constructing a graph \( G \) with \( O(n) \) edges such that \( G \to (P_n)_2 \). In the
last decades many successive improvements were obtained in order to determine
the size-Ramsey number of paths (see, e.g., [2, 3, 7, 20] for lower bounds, and
[2, 6, 7, 18] for upper bounds). The best known bounds for paths are \( 3n - 7 \leq \hat{r}_2(P_n) \leq 74n \).

For any \( s \geq 2 \) colours, Dudek and Prałat [7] and Krivelevich [17] proved that
there are constants \( c \) and \( C \) such that \( cs^2n \leq \hat{r}_s(P_n) \leq Cs^2(\log s)n \). Beck asked

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whether $\hat{r}_2(H)$ is linear for any bounded degree graph. This question was later answered negatively by R"{o}dl and Szemer"{e}di [19] who constructed a family $\{H_n\}_{n \in \mathbb{N}}$ of 3-regular graphs with $n$ vertices such that $\hat{r}_2(H_n) = \Omega(n \log^{1/60} n)$. The current best upper bound for the size-Ramsey number of bounded-degree graphs was obtained in [16] by Kohayakawa, R"{o}dl, Schacht and Szemer"{e}di, who proved that for any positive integer $\Delta$ there is a constant $c$ such that for any graph $H$ with $n$ vertices and maximum degree $\Delta$:

$$\hat{r}_2(H) \leq cn^{2-1/\Delta} \log^{1/\Delta} n.$$ 

For more results on the size-Ramsey number of bounded degree graphs see [5, 10, 12, 13, 14, 15].

Let us turn our attention to powers of bounded degree graphs. Let $H$ be a graph with $n$ vertices and let $k$ be a positive integer. The $k$th power $H^k$ of $H$ is the graph with vertex set $V(H)$ in which there is an edge between distinct vertices $u$ and $v$ if and only if $u$ and $v$ are at distance at most $k$ in $H$. Recently it was proved that the 2-colour size-Ramsey number of powers of paths and cycles is linear [4]. This result was extended to any fixed number $s$ of colours in [11], i.e.,

$$\hat{r}_s(P_n^k) = O_{k,s}(n) \quad \text{and} \quad \hat{r}_s(C_n^k) = O_{k,s}(n).$$

In our main result (Theorem 1) we generalize this result by proving that, for any positive integers $k$ and $s$, the $s$-colour size-Ramsey number of the $k$th power of any $n$-vertex bounded degree tree is linear in $n$.

**Theorem 1.** For any positive integers $k$, $\Delta$ and $s$ and any $n$-vertex tree $T$ with $\Delta(T) \leq \Delta$, we have

$$\hat{r}_s(T^k) = O_{k,\Delta,s}(n).$$

In Section 2 we give some auxiliary results and state two main lemmas used in the proof. A sketch of the proof of Theorem 1 is given in Section 3.

2. Auxiliary results

A graph $G$ is said to be $(n,a,b)$-expanding if for all $X \subset V(G)$ with $|X| \leq a(n-1)$, we have $|N_G(X)| \geq b|X|$. In the proof of our main result, we follow the main strategy developed in [11], combined with two new novel ingredients: (i) a result that states that any sufficiently large graph $G$ either contains a large expanding subgraph, or there is a reasonably balanced partition, into a given number of parts, of a large subset of $V(G)$ with no edges between any two parts; (ii) an embedding result that says that, to embed a power $T^k$ of a tree $T$ in a certain blow-up of a graph $G$, it is enough to find an embedding of some tree $T'$ in $G$. Results (i) and (ii) are given in their precise form in Lemmas 5 and 6.

The following embedding result due to Friedman and Pippenger [10] guarantees the existence of copies of bounded degree trees in expanding graphs.

**Lemma 2.** Let $n$ and $\Delta$ be positive integers and $G$ a non-empty graph. If $G$ is $(n,2,\Delta+1)$-expanding, then $G$ contains any $n$-vertex tree with maximum degree $\Delta$ as a subgraph.
Because of Lemma 2, we are interested in graph properties that guarantee expansion. One such property is ‘bijumbledness’. A graph $G$ on $N$ vertices is $(p, \alpha)$-bijumbled if, for all disjoint sets $X$ and $Y \subseteq V(G)$ with $|X| \leq |Y| \leq pN|X|$, we have $|e_G(X, Y) - p|X||Y|| \leq \alpha \sqrt{pN|X||Y|}$. Here, $e_G(X, Y)$ is the number of edges between $X$ and $Y$ in $G$.

**Lemma 3** (Bijumbledness implies expansion). For any positive $c$, $f$, $D$ and $\theta$ there is $c \geq 3$ such that the following holds. If $G$ is a graph on $|X|$ vertices that is $(c/n, \theta)$-bijumbled, then there exists a non-empty subgraph $H$ of $G$ that is $(n, f, D)$-expanding.

The following definition plays an important role in our proof.

**Definition 4.** For a positive number $n$ and positive numbers $a$, $b$, $\ell$, $\theta$, let $\mathcal{P}_n(a, b, \ell, \theta)$ denote the class of all graphs $G$ with the following properties, where $N = an$ and $p = c/N$.

(i) $|V(G)| = N$,

(ii) $\Delta(G) \leq b$,

(iii) $G$ has no cycles of length at most $2\ell$,

(iv) $G$ is $(p, \alpha)$-bijumbled.

We now state the main two novel ingredients in the proof of our main result, Theorem 1.

**Lemma 5.** For any numbers $f$, $D$, $\ell$ and $\eta$ there exists $A = (\ell - 1)(D + 1)$ such that the following holds for any sufficiently large $n$ and any graph $G$ on at least $An$ vertices:

(i) Either there is $\emptyset \neq Z \subseteq V(G)$ such that $G[Z]$ is $(n, f, D)$-expanding,

(ii) or there exist $V_1, \ldots, V_\ell \subseteq V(G)$ such that $|V_i| \geq \eta n$ for $1 \leq i \leq \ell$ and $G[V_i, V_j]$ is empty for any $1 \leq i < j \leq \ell$.

Let $G$ be a graph and $\ell \geq r$ be integers. An $(\ell, r)$-blow-up of $G$ is a graph obtained from $G$ by replacing every vertex of $G$ by a clique of size $\ell$ and for every edge of $G$ arbitrarily adding a complete bipartite graph $K_{r, r}$ between the two cliques corresponding to the vertices.

**Lemma 6** (Embedding lemma for powers of trees). For any positive integers $\Delta$ and $k$ there exist positive integers $r$ and $\ell_0$ such that the following holds for every $n$-vertex tree $T$ with maximum degree $\Delta$ and $\ell \geq \ell_0$. There exists a tree $T' = T'(T, \Delta, k)$ of maximum degree $\Delta^{2k}$ with at most $n + 1$ vertices such that for every graph $J$, if $T' \subseteq J$, then $T^{2k} \subseteq J'$ for any $(\ell, r)$-blow-up $J'$ of $J$.

3. **Proof of the main result**

We derive Theorem 1 from Proposition 7 below. But first, given an integer $\ell \geq 1$, let us define what we mean by a sheared complete blow-up $H(\ell)$ of a graph $H$: this is any graph obtained by replacing each vertex $v$ in $V(H)$ by a complete graph $C(v)$ with $\ell$ vertices, and by adding all edges but a perfect matching between $C(u)$ and $C(v)$ for each $uv \in E(H)$.
Proposition 7. For all integers \( k \geq 1, \Delta \geq 2, \) and \( s \geq 1 \) there exist positive reals \( r_s, a_s, b_s, c_s, \ell_s, \) and \( \theta_s \) for which the following holds. If \( n \) is sufficiently large and \( G \in \mathbb{P}_n(a_s, b_s, c_s, \ell_s, \theta_s) \) then, for any tree \( T \) on \( n \) vertices with \( \Delta(T) \leq \Delta, \) we have
\[
G^{r_s}\{\ell_s\} \rightarrow (T^k)_s.
\]

Theorem 1 follows from Proposition 7 applied to a certain subgraph of a random graph.

Proof of Theorem 1. Fix positive integers \( k, \Delta \) and \( s \) and let \( T \) be an \( n \)-vertex tree with maximum degree at most \( \Delta. \) Proposition 7 applied with parameters \( k, \Delta \) and \( s \) gives positive reals \( r_s, a_s, b_s, c_s, \ell_s, \) and \( \theta_s. \) Let \( N = 3a_s n. \) By considering a certain subgraph of the binomial random graph \( G(N, p) \) with \( p = c_s / N, \) one can show that there is a graph \( G \in \mathbb{P}_n(a_s, b_s, c_s, \ell_s, \theta_s), \) provided that \( n \) is sufficiently large. Proposition 7 tells us that \( G^{r_s}\{\ell_s\} \rightarrow (T^k)_s. \) Since \( |V(G)| = a_s n, \) \( \Delta(G) \leq b_s, \) and \( r_s \) and \( \ell_s \) are constants, we have \( |E(G^{r_s}\{\ell_s\})| = O_{k, \Delta, s}(n) \), which concludes the proof of Theorem 1. \( \square \)

We close with a sketch of the proof of Proposition 7. This proof is by induction on the number of colours \( s, \) and is based on Lemmas 8 and 9. Note that in the following there are some necessary conditions between the parameters \( a, b, c, \ell, \theta, \Delta \) and \( k \) that we omit for simplicity of this sketch.

Lemma 8 (Base Case). For all integers \( s \geq 1, k \geq 1 \) and \( \Delta \geq 2 \) there are positive \( a, b, c, \ell, \theta \) such that if \( n \) is sufficiently large, then the following holds for any \( G \in \mathbb{P}_n(a, b, c, \ell, \theta). \) For any \( n \)-vertex tree \( T \) with \( \Delta(T) \leq \Delta, \) the graph \( G^k\{\ell\} \) contains a copy of \( T^k. \)

Sketch of the proof of Lemma 8. We first note that, as \( G \) is bijumbled, Lemma 3 guarantees that \( G \) is expanded. Then, by Lemma 2, we see that there is a copy of \( T \) in \( G, \) which implies the existence of a copy of \( T^k \) in \( G^k. \) Finally, a greedy argument can be used to show that there is a copy of \( T^k \) in \( G^k\{\ell\}. \) \( \square \)

Lemma 9 (Induction Step). For any positive integers \( \Delta \geq 2, s \geq 2, \) \( k, r \) and positive reals \( a, b, c, \) and \( \ell \) and a sufficiently large constant \( \theta, \) there exist a positive integer \( r \) and positive reals \( a', b', c', \ell', \) and \( \theta' \) such that the following holds. If \( n \) is sufficiently large then for any graph \( G \in \mathbb{P}_n(a', b', c', \ell', \theta') \) and any \( s \)-colouring \( \chi \) of \( E(G^r\{\ell'\}) \) either

(i) there is a monochromatic copy of \( T^k \) in \( G^r\{\ell'\} \) for any \( n \)-vertex tree \( T \) with \( \Delta(T) \leq \Delta, \) or
(ii) there is \( H \in \mathbb{P}_n(a, b, c, \ell, \theta) \) such that \( H^r\{\ell\} \subset G^r\{\ell'\} \) and \( H^r\{\ell\} \) is coloured with at most \( s - 1 \) colours under \( \chi. \)

Sketch of the proof of Lemma 9. We start by fixing suitable constants \( r, a', b', c', \ell' \) and \( \theta'. \) Let \( n \) be sufficiently large and let \( G \in \mathbb{P}_n(a', b', c', \ell', \theta') \) be given. Give an arbitrary colouring \( \chi \) to the edges of a sheared complete blow-up \( G^r\{\ell'\} \) of \( G^r \) with \( s \) colours. We shall prove that either there is a monochromatic copy of \( T^k \) in \( G^r\{\ell'\}, \) or there is a graph \( H \in \mathbb{P}_n(a, b, c, \ell, \theta) \) such that a sheared
complete blow-up $H^r\{\ell\}$ of $H^r$ is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under $\chi$.

First, note that, by Ramsey’s theorem, if $\ell'$ is large then each $\ell'$-clique $C(v)$ of $G^{r'}\{\ell'\}$ contains a large monochromatic clique. Let blue be the colour of these monochromatic cliques in the majority of the $C(v)$. Let these blue cliques be $B(v) \subset C(v)$. Then we consider a graph $J \subset G^{r'}$ induced by the vertices $v$ corresponding to the blue cliques $B(v)$ and having only the edges $\{u, v\}$ such that there is a blue copy of $K_{r', \ell'}$ under $\chi$ in the bipartite graph induced between the blue cliques $B(u)$ and $B(v)$ in $G^{r'}\{L\}$.

Then, by Lemma 5 applied to $J$, either there is a set $\emptyset \neq W \subset V(J)$ such that $J[W]$ is expading, or there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in $J$. In the first case, Lemma 6 guarantees that there is a tree $T'$ such that, if $T' \subset J[W]$, then there is a blue copy of $T^k$ in $G^{r'}\{\ell'\}$. To prove that $T' \subset J[W]$, we recall that $J[W]$ is expanding and use Lemma 2. This finishes the proof of the first case.

Now let us consider the second case, in which there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in $J$. The idea is to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subset G^{r'}\{\ell'\}$ and, moreover, $H^r\{\ell\}$ does not have any blue edge. For that we first obtain a path $Q$ in $G$ with vertices $(x_1, \ldots, x_{2an})$ such that $x_i \in V_j$ for all $i$ where $i = j \mod \ell$. Then we partition $Q$ into $2an$ paths $Q_1, \ldots, Q_{2an}$ with $\ell$ vertices each, and consider an auxiliary graph $H'$ on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ with $Q_iQ_j \in E(H')$ if and only if $E_G(V(Q_i), V(Q_j)) \neq \emptyset$. We obtain a sparse subgraph $H'' \subset H'$ by choosing edges of $H'$ uniformly at random with a suitable probability $p$. Then, successively removing vertices of high degree, we obtain a graph $H \subset H''$ with $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$. It now remains to find a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. To do so, we first observe that the paths $Q_i \in V(H')$ give rise to $\ell'$-cliques in $G^{r'}\{r' \geq \ell\}$. One can then prove that there is a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges by applying the Lovász local lemma.

\[ \square \]

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