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THE SIZE-RAMSEY NUMBER OF POWERS OF BOUNDED DEGREE TREES

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Abstract. Given an integer $s \geq 1$, the *s-colour size-Ramsey number* of a graph H is the smallest integer m such that there exists a graph G with m edges with the property that, in any colouring of $E(G)$ with s colours, there is a monochromatic copy of H. We prove that, for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n.

1. Introduction

Given graphs G and H, and a positive integer s, we denote by $G \to (H)_s$ the property that any s-colouring of the edges of G contains a monochromatic copy of H. We are interested in the problem proposed by Erdős, Faudree, Rousseau and Schelp [**9**] of determining the minimum integer m for which there is a graph G with m edges such that property $G \to (H)_2$ holds. Formally, the *s-colour size-Ramsey number* $\hat{r}_s(H)$ of a graph H is defined as follows:

$$\hat{r}_s(H) := \min\{|E(G)|\colon G \to (H)_s\}.$$

Answering a question posed by Erdős [**8**], Beck [**2**] showed that $\hat{r}_2(P_n) = O(n)$ by means of a probabilistic proof. Alon and Chung [**1**] proved the same fact by explicitly constructing a graph G with $O(n)$ edges such that $G \to (P_n)_2$. In the last decades many successive improvements were obtained in order to determine the size-Ramsey number of paths (see, e.g., [**2, 3, 7, 20**] for lower bounds, and [**2, 6, 7, 18**] for upper bounds). The best known bounds for paths are $3n - 7 \leq \hat{r}_2(P_n) \leq 74n$.

For any $s \geq 2$ colours, Dudek and Prałat [**7**] and Krivelevich [**17**] proved that there are constants c and C such that $cs^2 n \leq \hat{r}_s(P_n) \leq Cs^2(\log s)n$. Beck asked

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whether $\hat{r}_2(H)$ is linear for any bounded degree graph. This question was later answered negatively by Rödl and Szemerédi [**19**] who constructed a family $\{H_n\}_{n\in\mathbb{N}}$ of 3-regular graphs with n vertices such that $\hat{r}_2(H_n) = \Omega(n\log^{1/60} n)$. The current best upper bound for the size-Ramsey number of bounded-degree graphs was obtained in [**16**] by Kohayakawa, Rödl, Schacht and Szemerdi, who proved that for any positive integer Δ there is a constant c such that for any graph H with n vertices and maximum degree Δ:

$$\hat{r}_2(H) \le cn^{2-1/\Delta}\log^{1/\Delta} n.$$

For more results on the size-Ramsey number of bounded degree graphs see [**5, 10, 12, 13, 14, 15**].

Let us turn our attention to powers of bounded degree graphs. Let H be a graph with n vertices and let k be a positive integer. The *kth power* H^k of H is the graph with vertex set $V(H)$ in which there is an edge between distinct vertices u and v if and only if u and v are at distance at most k in H. Recently it was proved that the 2-colour size-Ramsey number of powers of paths and cycles is linear [**4**]. This result was extended to any fixed number s of colours in [**11**], i.e.,

$$\hat{r}_s(P_n^k) = O_{k,s}(n) \quad \text{and} \quad \hat{r}_s(C_n^k) = O_{k,s}(n).$$

In our main result (Theorem 1) we generalize this result by proving that, for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n.

Theorem 1. *For any positive integers k, Δ and s and any n-vertex tree T with $\Delta(T) \le \Delta$, we have*

$$\hat{r}_s(T^k) = O_{k,\Delta,s}(n).$$

In Section 2 we give some auxiliary results and state two main lemmas used in the proof. A sketch of the proof of Theorem 1 is given in Section 3.

2. Auxiliary results

A graph G is said to be (n, a, b)-*expanding* if for all $X \subset V(G)$ with $|X| \le a(n-1)$, we have $|N_G(X)| \ge b|X|$. In the proof of our main result, we follow the main strategy developed in [**11**], combined with two new novel ingredients: (i) a result that states that any sufficiently large graph G either contains a large expanding subgraph, or there is a reasonably balanced partition, into a given number of parts, of a large subset of $V(G)$ with no edges between any two parts; (ii) an embedding result that says that, to embed a power T^k of a tree T in a certain blow-up of a graph G, it is enough to find an embedding of some tree T' in G. Results (i) and (ii) are given in their precise form in Lemmas 5 and 6.

The following embedding result due to Friedman and Pippenger [**10**] guarantees the existence of copies of bounded degree trees in expanding graphs.

Lemma 2. *Let n and Δ be positive integers and G a non-empty graph. If G is $(n, 2, \Delta+1)$-expanding, then G contains any n-vertex tree with maximum degree Δ as a subgraph.*

Because of Lemma 2, we are interested in graph properties that guarantee expansion. One such property is 'bijumbledness'. A graph G on N vertices is (p,α)-*bijumbled* if, for all disjoint sets X and $Y \subset V(G)$ with $|X| \leq |Y| \leq pN|X|$, we have $\big|e_G(X,Y) - p|X||Y|\big| \leq \alpha\sqrt{pN|X||Y|}$. Here, $e_G(X,Y)$ is the number of edges between X and Y in G.

Lemma 3 (Bijumbledness implies expansion)**.** *For any positive c, f, D and θ there is $a \geq 3$ such that the following holds. If G is a graph on an vertices that is $(c/n,\theta)$-bijumbled, then there exists a non-empty subgraph H of G that is (n,f,D)-expanding.*

The following definition plays an important role in our proof.

Definition 4. For a positive number n and positive numbers a, b, c, ℓ, θ, let $\mathcal{P}_n(a,b,c,\ell,\theta)$ denote the class of all graphs G with the following properties, where $N = an$ and $p = c/N$.

(i) $|V(G)| = N$,
(ii) $\Delta(G) \leq b$,
(iii) G has no cycles of length at most 2ℓ,
(iv) G is (p,θ)-bijumbled.

We now state the main two novel ingredients in the proof of our main result, Theorem 1.

Lemma 5. *For any numbers f, D, ℓ and η there exists $A = (\ell-1)(D+1)(\eta+f)+\eta$ such that the following holds for any sufficiently large n and any graph G on at least An vertices:*

(i) *Either there is $\emptyset \neq Z \subset V(G)$ such that $G[Z]$ is (n,f,D)-expanding,*
(ii) *or there exist $V_1,\dots,V_\ell \subseteq V(G)$ such that $|V_i| \geq \eta n$ for $1 \leq i \leq \ell$ and $G[V_i,V_j]$ is empty for any $1 \leq i < j \leq \ell$.*

Let G be a graph and $\ell \geq r$ be integers. An (ℓ,r)-*blow-up* of G is a graph obtained from G by replacing every vertex of G by a clique of size ℓ and for every edge of G arbitrarily adding a complete bipartite graph $K_{r,r}$ between the two cliques corresponding to the vertices.

Lemma 6 (Embedding lemma for powers of trees)**.** *For any positive integers Δ and k there exist positive integers r and ℓ_0 such that the following holds for every n-vertex tree T with maximum degree Δ and $\ell \geq \ell_0$. There exists a tree $T' = T'(T,\Delta,k)$ of maximum degree Δ^{2k} with at most $n+1$ vertices such that for every graph J, if $T' \subset J$, then $T^k \subset J'$ for any (ℓ,r)-blow-up J' of J.*

3. Proof of the main result

We derive Theorem 1 from Proposition 7 below. But first, given an integer $\ell \geq 1$, let us define what we mean by a *sheared complete blow-up* $H\{\ell\}$ of a graph H: this is any graph obtained by replacing each vertex v in $V(H)$ by a complete graph $C(v)$ with ℓ vertices, and by adding all edges *but a perfect matching* between $C(u)$ and $C(v)$ for each $uv \in E(H)$.

Proposition 7. *For all integers $k \geq 1$, $\Delta \geq 2$, and $s \geq 1$ there exist positive reals $r_s, a_s, b_s, c_s, \ell_s$ and θ_s for which the following holds. If n is sufficiently large and $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \theta_s)$ then, for any tree T on n vertices with $\Delta(T) \leq \Delta$, we have*

$$G^{r_s}\{\ell_s\} \to (T^k)_s.$$

Theorem 1 follows from Proposition 7 applied to a certain subgraph of a random graph.

Proof of Theorem 1. Fix positive integers k, Δ and s and let T be an n-vertex tree with maximum degree at most Δ. Proposition 7 applied with parameters k, Δ and s gives positive reals $r_s, a_s, b_s, c_s, \ell_s$ and θ_s. Let $N = 3a_s n$. By considering a certain subgraph of the binomial random graph $G(N,p)$ with $p = c_s/N$, one can show that there is a graph $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \theta_s)$, provided that n is sufficiently large. Proposition 7 tells us that $G^{r_s}\{\ell_s\} \to (T^k)_s$. Since $|V(G)| = a_s n$, $\Delta(G) \leq b_s$, and r_s and ℓ_s are constants, we have $|E(G^{r_s}\{\ell_s\})| = O_{k,\Delta,s}(n)$, which concludes the proof of Theorem 1. □

We close with a sketch of the proof of Proposition 7. This proof is by induction on the number of colours s, and is based on Lemmas 8 and 9. Note that in the following there are some necessary conditions between the parameters $a, b, c, \ell, \theta, \Delta$ and k that we omit for simplicity of this sketch.

Lemma 8 (Base Case)**.** *For all integers $s \geq 1$, $k \geq 1$ and $\Delta \geq 2$ there are positive a, b, c, ℓ, θ such that if n is sufficiently large, then the following holds for any $G \in \mathcal{P}_n(a, b, c, \ell, \theta)$. For any n-vertex tree T with $\Delta(T) \leq \Delta$, the graph $G^k\{\ell\}$ contains a copy of T^k.*

Sketch of the proof of Lemma 8. We first note that, as G is bijumbled, Lemma 3 guarantees that G is expanding. Then, by Lemma 2, we see that there is a copy of T in G, which implies the existence of a copy of T^k in G^k. Finally, a greedy argument can be used to show that there is a copy of T^k in $G^k\{\ell\}$. □

Lemma 9 (Induction Step)**.** *For any positive integers $\Delta \geq 2$, $s \geq 2$, k, r and positive reals a, b, c, and ℓ and a sufficiently large constant θ, there exist a positive integer r' and positive reals a', b', c', ℓ' and θ' such that the following holds. If n is sufficiently large then for any graph $G \in \mathcal{P}_n(a', b', c', \ell', \theta')$ and any s-colouring χ of $E(G^{r'}\{\ell'\})$ either*

- (i) *there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$ for any n-vertex tree T with $\Delta(T) \leq \Delta$, or*
- (ii) *there is $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subset G^{r'}\{\ell'\}$ and $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.*

Sketch of the proof of Lemma 9. We start by fixing suitable constants r', a', b', c', ℓ' and θ'. Let n be sufficiently large and let $G \in \mathcal{P}_n(a', b', c', \ell', \theta')$ be given. Give an arbitrary colouring χ to the edges of a sheared complete blow-up $G^{r'}\{\ell'\}$ of $G^{r'}$ with s colours. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

First, note that, by Ramsey's theorem, if ℓ' is large then each ℓ'-clique $C(v)$ of $G^{r'}\{\ell'\}$ contains a large monochromatic clique. Let blue be the colour of these monochromatic cliques in the majority of the $C(v)$. Let these blue cliques be $B(v) \subset C(v)$. Then we consider a graph $J \subset G^{r'}$ induced by the vertices v corresponding to the blue cliques $B(v)$ and having only the edges $\{u, v\}$ such that there is a blue copy of $K_{r',r'}$ under χ in the bipartite graph induced between the blue cliques $B(u)$ and $B(v)$ in $G^{r'}\{L\}$.

Then, by Lemma 5 applied to J, either there is a set $\emptyset \neq W \subset V(J)$ such that $J[W]$ is expading, or there are large disjoint sets $V_1, \dots, V_\ell$ with no edges between them in J. In the first case, Lemma 6 guarantees that there is a tree T' such that, if $T' \subset J[W]$, then there is a blue copy of T^k in $G^{r'}\{\ell'\}$. To prove that $T' \subset J[W]$, we recall that $J[W]$ is expanding and use Lemma 2. This finishes the proof of the first case.

Now let us consider the second case, in which there are large disjoint sets $V_1, \dots, V_\ell$ with no edges between them in J. The idea is to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subset G^{r'}\{\ell'\}$ and, moreover, $H^r\{\ell\}$ does not have any blue edge. For that we first obtain a path Q in G with vertices $(x_1, \dots, x_{2a\ell n})$ such that $x_i \in V_j$ for all i where $i = j \bmod \ell$. Then we partition Q into $2an$ paths $Q_1, \dots, Q_{2an}$ with ℓ vertices each, and consider an auxiliary graph H' on $V(H') = \{Q_1, \dots, Q_{2an}\}$ with $Q_iQ_j \in E(H')$ if and only $E_G(V(Q_i), V(Q_j)) \neq \emptyset$. We obtain a sparse subgraph $H'' \subset H'$ by choosing edges of H' uniformly at random with a suitable probability p. Then, successively removing vertices of high degree, we obtain a graph $H \subset H''$ with $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$. It now remains to find a copy of $H^r\{t\}$ in $G^{r'}\{\ell'\}$ with no blue edges. To do so, we first observe that the paths $Q_i \in V(H')$ give rise to ℓ-cliques in $G^{r'}$ ($r' \geq \ell$). One can then prove that there is a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges by applying the Lovász local lemma. $\square$

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