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A NEW LOWER BOUND ON HADWIGER-DEBRUNNER NUMBERS IN THE PLANE

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Abstract. A family of sets $\mathcal{F}$ is said to satisfy the (p,q)-property if among any p sets in $\mathcal{F}$ some q have a non-empty intersection. Hadwiger and Debrunner (1957) conjectured that for any $p \geq q \geq d+1$ there exists $c = c_d(p,q)$, such that any family of compact convex sets in $\mathcal{R}^d$ that satisfies the (p,q)-property can be pierced by at most c points. In a celebrated result from 1992, Alon and Kleitman proved the conjecture. However, obtaining sharp bounds on $c_d(p,q)$, known as the 'the Hadwiger-Debrunner numbers,' is still a major open problem in combinatorial geometry. The best currently known lower bound on the Hadwiger-Debrunner numbers in the plane is $c_2(p,q) = \Omega(\frac{p}{q}\log(\frac{p}{q}))$, while the best known upper bound is $O(p^{(1.5+\delta)(1+\frac{1}{q-2})})$.

In this paper we improve the lower bound significantly by showing that $c_2(p,q) \geq p^{1+\Omega(1/q)}$. Furthermore, the bound is obtained by a family of lines and is tight for all families that have a bounded VC-dimension. Unlike previous bounds on the Hadwiger-Debrunner numbers, which mainly used the weak epsilon-net theorem, our bound stems from a surprising connection of the (p,q)-problem to an old problem of Erdős on points in general position in the plane. We use a novel construction for the Erdős' problem, obtained recently by Balogh and Solymosi using the *hypergraph container method*, to get the lower bound on $c_2(p,3)$. We then generalize the bound to $c_2(p,q)$ for any $q \geq 3$.

1. Introduction

Helly's theorem, the (p,q)-theorem, and Hadwiger-Debrunner numbers. The classical Helly's theorem asserts that if in some finite family $\mathcal{F}$ of convex sets in $\mathcal{R}^d$, any $d+1$ sets have a non-empty intersection, then the whole family has a non-empty intersection, i.e., it can be *pierced* by one point. One of the most challenging extensions of Helly's theorem was introduced by relaxing the intersection assumption into a weaker assumption known as the (p,q)-property: Among any p sets in $\mathcal{F}$, some q have a non-empty intersection.

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Clearly, not every family that satisfies the (p, q)-property has a non-empty intersection; still, one may hope that such a family can be pierced by a 'small' number of points. Indeed, Hadwiger and Debrunner [**13**] conjectured that for all $p \geq q \geq d+1$, any family of convex sets in $\mathcal{R}^d$ that satisfies the (p, q)-property can be pierced by a constant number of points, independently of the size of the family. The minimum number of such points is denoted by $c = c_d(p, q)$. Hadwiger and Debrunner proved their conjecture for the special case when $q > \frac{d-1}{d}p + 1$, with $c = p - q + 1$; on the other hand, they showed that $p - q + 1$ is a lower bound on $c_d(p, q)$ for all pairs $p \geq q$.

After 35 years, the Hadwiger-Debrunner conjecture was proved in a celebrated result of Alon and Kleitman [**2**], also known as the (p, q)-theorem. The (p, q)-theorem has become a classical result in combinatorial geometry, and was generalized to various settings in numerous works (see, e.g., the survey [**10**]). In addition to its importance within combinatorics, it has found applications to diverse fields, including model theory in mathematical logic (see [**8**]) and social choice theory in economics (see [**5**]).

The upper bound on $c_d(p, q)$ yielded by the proof of the (p, q)-theorem is $\tilde{O}(p^{d^2+d})$ (for the case $q = d + 1$). Alon and Kleitman noted that this bound is far from being tight, and since then, the problem of obtaining tight bounds on $c_d(p, q)$ (also called the 'Hadwiger-Debrunner numbers' and denoted $HD_d(p, q)$) remains a major open problem in combinatorial geometry.

Despite extensive research, little is known about the asymptotics of $HD_d(p, q)$. Near optimal upper bounds were very recently obtained for very large values of q (for example, $HD_d(p, q) \leq p - q + 2$ for all $q > p^{\frac{d-1}{d}+\epsilon}$ [**18**]). Tight bounds were also obtained for specific classes of families (e.g., families of axis-parallel rectangles, see [**9, 16**]), and for specific values of p, q, d (see [**15**]). However, neither of these results extends to general (p, q).

Weak epsilon-nets and their relation to $HD_d(p, q)$. The best currently known lower bounds on $HD_d(p, q)$ are obtained by lower bounds on the so-called *weak epsilon-nets*. For a finite family of points $\mathcal{G} \subset \mathcal{R}^d$ and for $\epsilon > 0$, a weak ϵ-net for $\mathcal{G}$ is a set S of points (not necessarily in $\mathcal{G}$) such that any convex set $T \subset \mathcal{R}^d$ that contains at least $\epsilon|\mathcal{G}|$ points of $\mathcal{G}$ also contains a point of S.

Alon et al. [**1**] proved that for any d, ϵ there exists a bound $f_d(\epsilon)$ such that any finite $\mathcal{G} \subset \mathcal{R}^d$ admits a weak ϵ-net of size at most $f_d(\epsilon)$. However, the bound on $f_d(\epsilon)$ is far from being tight, and improving it remains another important open problem. In a very recent breakthrough, Rubin [**20**] showed that for any $\delta > 0$, every $\mathcal{G} \subset \mathcal{R}^2$ admits a weak ϵ-net of size at most $O_\delta(\epsilon^{-1.5-\delta})$. This is still far from the best known lower bound $f_d(\epsilon) = \frac{1}{\epsilon}\log^{d-1}(\frac{1}{\epsilon})$ obtained by Bukh, Matoušek and Nivasch [**7**], which is conjectured to be close to tight.

Weak ϵ-nets are closely related to the (p, q)-theorem. Indeed, for any set of points $\mathcal{G}$, it is easy to see that the family $\mathcal{F}$ of all convex sets that contain at least $\epsilon_0 = q/p$ points of $\mathcal{G}$ satisfies the (p, q)-property. If the size of the smallest weak ϵ_0-net for $\mathcal{G}$ is ℓ, then $\mathcal{F}$ is a family of convex sets that satisfies the (p, q)-property and cannot be pierced by less than ℓ points. Therefore, any lower bound on $f_d(\epsilon)$

translates immediately into a lower bound on $HD_d(p,q)$. The best known lower bound on $HD_d(p,q)$ is of the form:

$$HD_d(p,q) = \Omega\left(\frac{p}{q}\log^{d-1}\left(\frac{p}{q}\right)\right), \tag{1}$$

following immediately from the aforementioned lower bound of Bukh et al. [**7**] on $f_d(\epsilon)$.

While upper bounds on $f_d(\epsilon)$ do not translate directly into upper bounds on $HD_d(p,q)$, the weak epsilon-net theorem plays a central role in the Alon-Kleitman's proof of the (p,q)-theorem, and the best currently known general upper bound for $HD_d(p,q)$, obtained in [**18**, Proposition 2.6], is formulated in terms of $f_d(\epsilon)$:

$$HD_d(p,q) \leq f_d\left(\Omega(p^{-1-\frac{d-1}{q-d}})\right). \tag{2}$$

In particular, using Rubin's result [**20**], in the plane we have

$$HD_2(p,q) = O(p^{(1.5+\delta)(1+\frac{1}{q-2})})$$

for any $\delta > 0$ and $p > p_0(\delta)$.

2. Our main result

In this paper we present a new general lower bound on $HD_d(p,q)$ which is a significant improvement over the best previous bound $HD_d(p,q) = \Omega(\frac{p}{q}\log^{d-1}(\frac{p}{q}))$:

Theorem 2.1. *For any $0 < \eta < 1/2$ and for any $p,q \geq 3$ such that $q \leq 0.01\eta \cdot (\frac{\log p}{\log\log p})^{1/3}$, there exists a family $\mathcal{F}$ of lines in $\mathcal{R}^2$ that satisfies the (p,q)-property and cannot be pierced by less than $p^{1+\frac{1-\eta}{4q-7}}$ points. Consequently, $HD_d(p,q) \geq p^{1+\frac{1-\eta}{4q-7}}$ for all $d \geq 2$.*

Importantly, while our lower bound construction uses a family of lines, which are, in some sense, the 'simplest' convex objects, it is tight for a wide class of families, namely, all families whose so-called *VC-dimension* is bounded.

To explain the above statement, a few definitions are needed. For a family of sets $\mathcal{F}$, a set C is said to be *shattered* by $\mathcal{F}$ if the set $\{F \cap C \ : \ F \in \mathcal{F}\}$ contains all $2^{|C|}$ subsets of C. The VC-dimension of $\mathcal{F}$ is $\sup\{c \in \mathcal{N} : \mathcal{F}$ shatters some set of cardinality $c\}$. For example, it is easy to see that the VC-dimension of any family of lines is at most 2.

The notion 'VC-dimension' was introduced by Vapnik and Chervonenkis [**22**], and since then has found numerous applications (e.g., to computational geometry and to machine learning) and has been studied extensively in the past few decades (see, e.g., [**19**]). Haussler and Welzl [**14**] proved that any family $\mathcal{G}$ with VC-dimension at most r admits a weak ϵ-net (and actually, the significantly stronger notion of 'ϵ-net', see [**19**]) of size $O(\frac{r}{\epsilon}\log(\frac{r}{\epsilon}))$.

Substituting the assertion of the Haussler-Welzl theorem into (2), we obtain the upper bound $HD_2(p,q) \leq O\left(p^{1+\frac{1}{q-2}}\log p\right)$ for any finite family $\mathcal{F}$ of convex sets in the plane with a bounded VC-dimension. Therefore, Theorem 2.1 shows that within the class of families with a bounded VC-dimension we have $c_d(p,q) = p^{1+\Theta(1/q)}$.

Connection to a problem of Erdős on points in general position in the plane. While the best previously known bounds on the Hadwiger-Debrunner numbers were obtained via improved bounds for the weak epsilon-net theorem, our bound stems from a surprising connection between the (p,q)-problem and an old problem of Erdős regarding points in general position in the plane.

In [**11**], Erdős raised the following problem: What is the maximal possible $\ell = \ell(n)$ such that any set S of n points, with no four of them collinear, contains a subset of size ℓ in general position (that is, with no three collinear points)?

Until recently, the best known upper bound for Erdős problem was $\ell(n) = o(n)$, proved by Füredi [**12**] using the Density Hales-Jewett theorem of Katznelson and Furstenberg (1989). In a major breakthrough, Balogh and Solymosi [**6**] proved that $\ell(n) \leq n^{5/6+\delta}$, for any $\delta > 0$ and any $n > n_0(\delta)$.

The main observation underlying our results is that an upper bound for the Erdős problem is directly translated into a lower bound on $HD_2(p,3)$. Indeed, let S be a set of n points in the plane with no collinear 4-tuple, such that any subset of S of size at least $\ell(n)$ contains a collinear 3-tuple. By point-line duality in the plane, we can transform S into a family $\mathcal{F}$ of n lines, such that no four lines share a common point, while each subset of $\mathcal{F}$ of size $\ell(n)$ contains three lines with a common point. The latter condition means exactly that $\mathcal{F}$ satisfies the $(\ell(n),3)$ property. On the other hand, the former condition implies that $\mathcal{F}$ cannot be pierced by less than $n/3$ points. Hence, $\mathcal{F}$ is a family of convex sets in the plane that satisfies the $(\ell(n),3)$ property but cannot be pierced by less than $n/3$ points, and thus, $HD_2(\ell(n),3) \geq n/3$.

Combining this observation with the result of Balogh and Solymosi, we immediately obtain the lower bound

$$HD_2(p,3) \geq p^{\frac{6}{5}-\delta},$$

for all $\delta > 0$ and $p > p_0(\delta)$, which is the assertion of Theorem 2.1 in the case $q = 3$. The result for a general $q \geq 3$ is much more involved; it requires generalizing the construction of Balogh and Solymosi and their argument to random subsets of the hypergraph $\mathcal{H}(n, 2q-2, q)$ whose vertices are the points in the $(2q-2)$-dimensional grid and whose hyperedges are collinear q-tuples. Importantly, the choice of dimension is crucial for obtaining Theorem 2.1; applying the same technique with the q-dimensional grid leads to a significantly weaker result.

3. Main techniques and proof outline

Our proof follows and extends the proof framework of Balogh and Solymosi [**6**], whose heart is an application of the hypergraph container method.

The hypergraph container method. This method was introduced independently by Saxton and Thomason [**21**] and by Balogh, Morris, and Samotij [**3**]. Intuitively, for a hypergraph $H = (V,E)$ whose co-degrees are 'distributed evenly,' the method

allows us to find a relatively small family $\mathcal{C}$ of 'not-too-large' subsets of V known as 'containers,' such that each independent set in V is included in some container $C \in \mathcal{C}$. This, in turn, allows us to bound the number of independent sets of any fixed size. In the few years since the method was introduced, it has been applied to numerous problems in extremal graph theory, Ramsey theory, and additive combinatorics (see the survey [**4**]). The result of Balogh and Solymosi was the first application of the method to combinatorial geometry.
Proof outline. For $n, k, q \in \mathcal{N}$, we denote by $\mathcal{H}(n, k, q)$ the q-uniform hypergraph whose vertex set is the k-dimensional grid $\{1, 2, \ldots, n\}^k$, and whose hyperedges are all collinear q-tuples of vertices. The proof of our main theorem consists of three stages:

1. Reduction stage. We show that it is sufficient to prove that for some n, k, u, there exists a subset S of $\{1, 2, \ldots, n\}^k$ of size at least $(u-1) \cdot p^{1+\frac{1-\eta}{4q-7}}$ that does not contain collinear u-tuples and also does not contain independent sets of size at least p of the hypergraph $\mathcal{H}(n, k, q)$.

2. A general upper bound on the number of independent m-subsets of $\mathcal{H}(n, k, r)$. We obtain an upper bound on the number of independent subsets of size m of the hypergraph $\mathcal{H}(n, k, r)$, as a function of n, k, r, m. The idea behind this stage is that if the number of independent subsets of size p of $\mathcal{H}(n, k, q)$ is 'small', then it is easier for a randomly chosen subset of the vertices of $\mathcal{H}(n, k, q)$ to be free of independent sets of size p. This stage uses the *hypergraph container method.*

3. Probabilistic construction. We construct the required set S using the probabilistic method. Specifically, we consider an α-random subset $\tilde{S}$ of $\{1, 2, \ldots, n\}^k$ for some n, k, α. We show that for an appropriate choice of all the parameters involved, with a positive probability, the subset $\tilde{S}$ does not contain independent sets of $\mathcal{H}(n, k, q)$ of size p and contains only a small amount of collinear u-tuples, so that we can remove them and obtain a set S of size at least $(u-1) \cdot p^{1+\frac{1-\eta}{4q-7}}$ with no collinear u-tuples and no independent subsets of $\mathcal{H}(n, k, q)$ of size p.

The full proof, as well as an application of our technique to a natural hypergraph coloring problem, can be found in the full version of the paper available online [**17**].

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