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SOME RESULTS AROUND THE ERDŐS MATCHING CONJECTURE

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Abstract. More than 50 years ago, Erdős asked the following question: what is the largest family of k-element subsets of $[n]$ with no s pairwise disjoint sets? In this abstract, we discuss recent progress on this problem and its generalizations.

1. The Erdős Matching Conjecture

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family of k-element subsets on the vertex set $[n] := \{1, \ldots, n\}$. Erdős suggested the following problem: determine the maximum of $|\mathcal{F}|$, given that $\mathcal{F}$ has no s pairwise disjoint sets. Each of the following families satisfies this requirement.

$$\mathcal{A}_i := \left\{ A \in \binom{[n]}{k} : |A \cap [is-1]| \geq i \right\}. \tag{1}$$

Erdős Matching Conjecture (Erdős, [**5**])**.** If $n \geq k(s+1)$ and $\mathcal{F} \subset \binom{[n]}{k}$ has no s pairwise disjoint sets then

$$|\mathcal{F}| \leq \max\left\{|\mathcal{A}_1|, |\mathcal{A}_k|\right\} = \max\left\{\binom{n}{k} - \binom{n-s+1}{k}, \binom{ks-1}{k}\right\}. \tag{2}$$

The Erdős Matching Conjecture, or EMC for short, is trivial for $k = 1$ and was proved by Erdős and Gallai [**6**] for $k = 2$. It was settled in the case $k = 3$ [**20, 25, 9**]. The case $s = 2$ is the classical Erdős-Ko-Rado theorem [**7**] which was the starting point of a large part of ongoing research in extremal set theory.

In his original paper, Erdős proved (2) for $n \geq n_0(k, s)$. After some improvements [**4, 22**], the current best bound is due to the first author, who proved (2) for $n \geq 2sk - s$ (cf. [**8**]). An easy computation shows that $|\mathcal{A}_1| > |\mathcal{A}_k|$ already for $n \geq (k+1)s$, that is, $|\mathcal{F}| \leq \binom{n}{k} - \binom{n-s+1}{k}$ should hold also for $(k+1)s < n < 2sk - s$.

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For $n = ks$ the EMC was implicitly proved by Kleitman [**24**]. This was extended very recently by the first author [**10**], who showed that $|\mathcal{F}| \le |\mathcal{A}_k|$ in (2) for all $n \le s(k+\varepsilon)$, where ε depends on k. Our first result is the following theorem.

Theorem 1 ([**17**]). *There exists an absolute constant s_0, such that any $\mathcal{F} \subset \binom{[n]}{k}$ with no s pairwise disjoint sets satisfies*

$$|\mathcal{F}| \le \binom{n}{k} - \binom{n-s+1}{k}, \tag{3}$$

provided $n \ge \frac{5}{3}sk - \frac{2}{3}s$ and $s \ge s_0$.

Roughly speaking, Theorem 1 settles the EMC for 1/3 of the cases left over by [**8**]. We believe that the EMC is one the most important open problems in extremal set theory, playing a major role in several extremal problems in combinatorics and beyond (see, e.g., [**3**]). In particular, the EMC is related for the study of its non-uniform analogue, suggested by Erdős and Kleitman [**24**]. We have recently obtained significant progress on this and related questions [**11**]–[**16**].

2. Several families

We say that $\mathcal{F}_1, \dots, \mathcal{F}_s$ are cross-dependent if there are no sets $F_1 \in \mathcal{F}_1, \dots, F_s \in \mathcal{F}_s$ that are pairwise disjoint. The following multipartite version of the EMC was addressed by Aharoni and Howard [**1**], as well as by Huang, Loh and Sudakov [**22**]:

Problem 1. Given that the families $\mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ that are cross-dependent, find $\min_{i \in [s]} |\mathcal{F}_i|$.

We note here that some authors use the term "$\mathcal{F}_1, \dots, \mathcal{F}_s$ contain a rainbow matching" to refer to the situation, opposite to "cross-dependence". In [**22**], the authors proved the following result.

Theorem 2 ([**22**]). *If $n > 3sk^2$ and $\mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are cross-dependent then*

$$\min_{i \in [s]} |\mathcal{F}_i| \le \binom{n}{k} - \binom{n-s+1}{k}. \tag{4}$$

It is clear that the bound here is attained on $\mathcal{F}_1 = \dots = \mathcal{F}_s = \mathcal{A}_1$ and that, substituting $\mathcal{F} = \mathcal{F}_1 = \dots = \mathcal{F}_s$, one recovers the statement of the EMC. Unfortunately, the techniques developed in [**8**] and [**17**] do not seem to apply to the more general setting of Problem 1.

The bound (4) was obtained for $n > f(s)k$ with some unspecified and very fast growing function $f(s)$ by Keller and Lifshitz in [**23**] as an application of the junta method. We note that their results apply to a much more general setting. In [**18**], we managed to obtain sharp junta approximation-type results for shifted families.

Definition 1. Consider two sets $F_i = (a_1^i, \dots, a_k^i)$ with $a_1^i < a_2^i < \dots < a_k^i$ for $i = 1, 2$. Then $F_1 \prec_s F_2$ iff $a_j^1 \le a_j^2$ for every $j \in [k]$. We say that a family $\mathcal{F} \subset \binom{[n]}{k}$ is *shifted* if $F \in \mathcal{F}$ and $G \prec_s F$ implies $G \in \mathcal{F}$.

As a result, we managed to improve Theorem 2 to an almost-linear bound.

Theorem 3 ([**18**]). *The statement of Theorem 2 holds for* $n \geq 12ks\log(e^2 s)$.

We note that the validity of Theorem 3 for $n > Csk$ with some large C was announced by Keevash, Lifshitz, Long, and Minzer as a consequence of general sharp threshold-type results.

3. Beyond the Erdős Matching Conjecture

Let us introduce the following general notion.

Definition 1. Let $k, s \geq 2$ and $k \leq q < sk$ be integers. A k-graph $\mathcal{F} \subset \binom{[n]}{k}$ is said to have property $U(s,q)$ if

$$|F_1 \cup \cdots \cup F_s| \leq q \tag{5}$$

for all choices of $F_1, \ldots, F_s \in \mathcal{F}$. For shorthand, we will also say $\mathcal{F}$ is $U(s,q)$ to refer to this property.

Not that being $U(2, 2k-t)$ is equivalent to being t-intersecting[1] and, similarly, being $U(s, sk-1)$ is equivalent to having no s pairwise disjoint sets. Define the following families.

$$\mathcal{A}_{p,r} := \left\{ A \in \binom{[n]}{k} : |A \cap [p]| \geq r \right\}. \tag{6}$$

Note that $\mathcal{A}_{p,r}$ is $U(s, (k-r)s+p)$ for all s. Note also that, comparing this with (1), we have $\mathcal{A}_i = \mathcal{A}_{is-1,s}$. With this notation, the famous Complete Intersection Theorem [**2**] states that

$$\text{if } \mathcal{F} \text{ is } U(2, 2k-t) \text{ then } |\mathcal{F}| \leq \max_{0 \leq i \leq k-t} |\mathcal{A}_{2i+t,i+t}|, \tag{7}$$

and one may reformulate the EMC analogously:

$$\text{if } \mathcal{F} \text{ is } U(s, sk-1) \text{ then } |\mathcal{F}| \leq \max_{i \in \{1,k\}} |\mathcal{A}_{si-1,i}|. \tag{8}$$

Thus, the EMC and the Complete Intersection Theorem may essentially be seen as particular cases of the following general conjecture.

Conjecture 1. Fix n, k, s, q and assume that $\mathcal{F} \subset \binom{[n]}{k}$ is $U(s,q)$, where $q = (k-r)s + p$ with $r \leq p \leq s + r - 2$. Then $|\mathcal{F}| \leq \max_{0 \leq i \leq k-r} |\mathcal{A}_{p+is,r+i}|$.

(The statement of the EMC is actually slightly stronger, stating that $U(s, sk-1)$ is attained on of the two possible values of i rather than k, as suggested by the conjecture.)

We managed to verify the conjecture for a wide range of the parameters.

[1]That is, $|F_1 \cap F_2| \geq t$ for any $F_1, F_2 \in \mathcal{F}$.

Theorem 4 ([**19**])**.** *Fix some integers n, k, s, p, r, such that $1 \le r \le k$ and $r \le p \le s+r-2$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ has property $U(s,q)$ for $q = (k-r)s+p$. If $n \ge C(s,r)k$, then*[2]

$$|\mathcal{F}| \le |\mathcal{A}_{p,r}|.$$

For $r = p = 1$, the theorem (with the precise form of $C(s,r)$) implies the following.

Corollary 1. *For $n \ge s^2k$ any family $\mathcal{F} \subset \binom{[n]}{k}$ that is $UP(s,(k-1)s+1)$ has size at most $\binom{n-1}{k-1}$, which is the size of the largest intersecting family $\{F \in \binom{[n]}{k} : 1 \in F\}$.*

Thus, Theorem 4 can be seen as a sharpening of the Erdős–Ko–Rado theorem. Indeed, if a family $\mathcal{F}$ is intersecting, then the union of any s sets has size at most $(k-1)s+1$, but not vice versa.

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[2]The statement is simplified, but the function $C(s,r)$ is roughly s^{r+1}. Moreover, one recovers the bound from [**8**] for $q = sk-1$.

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