SOME RESULTS AROUND THE ERDŐS MATCHING CONJECTURE

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Abstract. More than 50 years ago, Erdős asked the following question: what is the largest family of $k$-element subsets of $[n]$ with no $s$ pairwise disjoint sets? In this abstract, we discuss recent progress on this problem and its generalizations.

1. The Erdős Matching Conjecture

Let $F \subseteq \binom{[n]}{k}$ be a family of $k$-element subsets on the vertex set $[n] := \{1, \ldots, n\}$. Erdős suggested the following problem: determine the maximum of $|F|$, given that $F$ has no $s$ pairwise disjoint sets. Each of the following families satisfies this requirement.

(1) $A_i := \left\{ A \in \binom{[n]}{k} : |A \cap [i - 1]| \geq i \right\}$.

Erdős Matching Conjecture (Erdős, [5]). If $n \geq k(s + 1)$ and $F \subseteq \binom{[n]}{k}$ has no $s$ pairwise disjoint sets then

(2) $|F| \leq \max \{|A_1|, |A_k|\} = \max \left\{ \binom{n}{k} - \binom{n - s + 1}{k}, \binom{ks - 1}{k} \right\}$.

The Erdős Matching Conjecture, or EMC for short, is trivial for $k = 1$ and was proved by Erdős and Gallai [6] for $k = 2$. It was settled in the case $k = 3$ [20, 25, 9]. The case $s = 2$ is the classical Erdős-Ko-Rado theorem [7] which was the starting point of a large part of ongoing research in extremal set theory.

In his original paper, Erdős proved (2) for $n \geq n_0(k, s)$. After some improvements [4, 22], the current best bound is due to the first author, who proved (2) for $n \geq 2sk - s$ (cf. [8]). An easy computation shows that $|A_1| > |A_k|$ already for $n \geq (k+1)s$, that is, $|F| \leq \binom{n}{k} - \binom{n-s+1}{k}$ should hold also for $(k+1)s < n < 2sk-s$.

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For \( n = ks \) the EMC was implicitly proved by Kleitman [24]. This was extended very recently by the first author [10], who showed that \(|F| \leq |A_k|\) in (2) for all \( n \leq s(k + \varepsilon) \), where \( \varepsilon \) depends on \( k \). Our first result is the following theorem.

**Theorem 1 ([17]).** There exists an absolute constant \( s_0 \), such that any \( F \subset \binom{[n]}{k} \) with no \( s \) pairwise disjoint sets satisfies

\[
|F| \leq \binom{n}{k} - \binom{n - s + 1}{k},
\]

provided \( n \geq \frac{5}{4}sk - \frac{2}{5}s \) and \( s \geq s_0 \).

Roughly speaking, Theorem 1 settles the EMC for \( 1/3 \) of the cases left over by [8]. We believe that the EMC is one the most important open problems in extremal set theory, playing a major role in several extremal problems in combinatorics and beyond (see, e.g., [3]). In particular, the EMC is related for the study of its non-uniform analogue, suggested by Erdős and Kleitman [24]. We have recently obtained significant progress on this and related questions [11]–[16].

2. **Several families**

We say that \( F_1, \ldots, F_s \) are cross-dependent if there are no sets \( F_1 \in F_1, \ldots, F_s \in F_s \) that are pairwise disjoint. The following multipartite version of the EMC was addressed by Aharoni and Howard [1], as well as by Huang, Loh and Sudakov [22]:

**Problem 1.** Given that the families \( F_1, \ldots, F_s \subset \binom{[n]}{k} \) that are cross-dependent, find \( \min_{i \in [s]} |F_i| \).

We note here that some authors use the term “\( F_1, \ldots, F_s \) contain a rainbow matching” to refer to the situation, opposite to “cross-dependence”. In [22], the authors proved the following result.

**Theorem 2 ([22]).** If \( n > 3sk^2 \) and \( F_1, \ldots, F_s \subset \binom{[n]}{k} \) are cross-dependent then

\[
\min_{i \in [s]} |F_i| \leq \binom{n}{k} - \binom{n - s + 1}{k}.
\]

It is clear that the bound here is attained on \( F_1 = \cdots = F_s = A_1 \) and that, substituting \( F = F_1 = \cdots = F_s \), one recovers the statement of the EMC. Unfortunately, the techniques developed in [8] and [17] do not seem to apply to the more general setting of Problem 1.

The bound (4) was obtained for \( n > f(s)k \) with some unspecified and very fast growing function \( f(s) \) by Keller and Lifshitz in [23] as an application of the junta method. We note that their results apply to a much more general setting. In [18], we managed to obtain sharp junta approximation-type results for shifted families.

**Definition 1.** Consider two sets \( F_1 = (a_1^i, \ldots, a_k^i) \) with \( a_1^i < a_2^i < \cdots < a_k^i \) for \( i = 1, 2 \). Then \( F_1 \prec_s F_2 \) if \( a_j^1 \leq a_j^2 \) for every \( j \in [k] \). We say that a family \( F \subset \binom{[n]}{k} \) is **shifted** if \( F \in \mathcal{F} \) and \( G \prec_s F \) implies \( G \in \mathcal{F} \).
As a result, we managed to improve Theorem 2 to an almost-linear bound.

**Theorem 3** ([18]). The statement of Theorem 2 holds for \( n \geq 12ks \log(e^2s) \).

We note that the validity of Theorem 3 for \( n > Csk \) with some large \( C \) was announced by Keevash, Lifshitz, Long, and Minzer as a consequence of general sharp threshold-type results.

3. Beyond the Erdős Matching Conjecture

Let us introduce the following general notion.

**Definition 1.** Let \( k, s \geq 2 \) and \( k \leq q < sk \) be integers. A \( k \)-graph \( F \subset \binom{[n]}{k} \) is said to have property \( U(s, q) \) if
\[
|F_1 \cup \cdots \cup F_s| \leq q
\]
for all choices of \( F_1, \ldots, F_s \in F \). For shorthand, we will also say \( F \) is \( U(s, q) \) to refer to this property.

Not that being \( U(2, 2k - t) \) is equivalent to being \( t \)-intersecting\(^1\) and, similarly, being \( U(s, sk - 1) \) is equivalent to having no \( s \) pairwise disjoint sets. Define the following families.

\[
\mathcal{A}_{p,r} := \left\{ A \in \binom{[n]}{k} : |A \cap [p]| \geq r \right\}.
\]

Note that \( \mathcal{A}_{p,r} \) is \( U(s, (k - r)s + p) \) for all \( s \). Note also that, comparing this with (1), we have \( A_i = \mathcal{A}_{is-1,s} \). With this notation, the famous Complete Intersection Theorem [2] states that
\[
\text{if } F \text{ is } U(2, 2k - t) \text{ then } |F| \leq \max_{0 \leq i \leq k-t} |\mathcal{A}_{2i+t,i+t}|,
\]
and one may reformulate the EMC analogously:
\[
\text{if } F \text{ is } U(s, sk - 1) \text{ then } |F| \leq \max_{i \in \{1,k\}} |\mathcal{A}_{si-1,i}|.
\]

Thus, the EMC and the Complete Intersection Theorem may essentially be seen as particular cases of the following general conjecture.

**Conjecture 1.** Fix \( n, k, s, q \) and assume that \( F \subset \binom{[n]}{k} \) is \( U(s, q) \), where \( q = (k - r)s + p \) with \( r \leq p \leq s + r - 2 \). Then \( |F| \leq \max_{0 \leq i \leq k-r} |\mathcal{A}_{p+i,s+r+i}| \).

(The statement of the EMC is actually slightly stronger, stating that \( U(s, sk - 1) \) is attained on of the two possible values of \( i \) rather than \( k \), as suggested by the conjecture.)

We managed to verify the conjecture for a wide range of the parameters.

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\(^1\)That is, \( |F_1 \cap F_2| \geq t \) for any \( F_1, F_2 \in F \).
Theorem 4 ([19]). Fix some integers \( n, k, s, p, r \), such that \( 1 \leq r \leq k \) and \( r \leq p \leq s + r - 2 \). Suppose that \( \mathcal{F} \subset \binom{[n]}{k} \) has property \( U(s, q) \) for \( q = (k - r)s + p \). If \( n \geq C(s, r)k \), then
\[
|\mathcal{F}| \leq |A_{p,r}|.
\]

For \( r = p = 1 \), the theorem (with the precise form of \( C(s, r) \)) implies the following.

Corollary 1. For \( n \geq s^2k \) any family \( \mathcal{F} \subset \binom{[n]}{k} \) that is \( UP(s, (k-1)s + 1) \) has size at most \( \binom{n-1}{k-1} \), which is the size of the largest intersecting family \( \{ F \in \binom{[n]}{k} : 1 \in F \} \).

Thus, Theorem 4 can be seen as a sharpening of the Erdős–Ko–Rado theorem. Indeed, if a family \( \mathcal{F} \) is intersecting, then the union of any \( s \) sets has size at most \( (k - 1)s + 1 \), but not vice versa.

References

1. Aharoni R. and Howard D., Size conditions for the existence of rainbow matchings, unpublished manuscript.

\(^2\)The statement is simplified, but the function \( C(s, r) \) is roughly \( s^{r+1} \). Moreover, one recovers the bound from [8] for \( q = sk - 1 \).

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