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CYCLES THROUGH A SET OF SPECIFIED VERTICES OF A PLANAR GRAPH

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Abstract. Confirming a conjecture of Plummer, Thomas and Yu proved that a 4-connected planar graph contains a cycle through all but two (freely choosable) vertices. Here we prove that a planar graph G contains a cycle through $X \setminus \{x_1, x_2\}$ if $X \subseteq V(G)$, X large enough, $x_1, x_2 \in X$, and X cannot be separated in G by removing less than 4 vertices.

In the present paper, we consider simple, finite, and undirected graphs G, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For graph terminology not defined here, we refer to [**1**].

Tutte [**8**] proved that every 4-connected planar graph has a hamiltonian cycle, and Thomassen [**7**] generalized this result by showing that every 4-connected planar graph has a hamiltonian path connecting every pair of two specified vertices. Eventually, Sanders [**5**] extended the results of Thomassen and of Tutte and proved the following

Theorem 1 (Sanders [**5**])**.** *Every 4-connected planar graph has a hamiltonian cycle containing any two specified edges.*

The results in [**5**] readily imply that the deletion of any vertex from a 4-connected planar graph results in a hamiltonian graph. Thomas and Yu [**6**] proved a conjecture of Plummer [**4**] that this is also true if two vertices are deleted.

Theorem 2 (Thomas, Yu [**6**])**.** *Let G be a 4-connected planar graph and $x, y \in V(G)$. Then $G - \{x, y\}$ has a hamiltonian cycle.*

It is well known, that for each integer k there is a 3-connected graph on n vertices such that a longest cycle contains at most $n - k$ vertices. Furthermore, if three vertices of a 4-separator of a graph are removed, then the resulting graph will not contain a hamiltonian cycle. Thus, Theorem 2 is best possible.

Assume that G is a 4-connected planar graph on the vertex set $V(G)$, thus, $|V(G)| \geq 6$. By Menger's Theorem [**3**], the 4-connectivity of G is equivalent to the fact that each pair of vertices is connected by at least four paths in G such that each two of these paths have just these two vertices as end vertices in common.

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By the previous Theorem 2, we know that G has a cycle C through all but two freely selectable vertices x and y.

Let H be a subdivision of G. Then H contains a subdivision C' of the cycle C containing $V(G) \setminus \{x, y\}$ and, moreover, there are at least four internally vertex-disjoint paths in H between any two vertices of $V(G)$. In this sense, $V(G)$ is still 4-connected in H, even through H is only a 2-connected graph and the existence of the cycle C' in H cannot guarantied by Theorem 2 anymore.

Motivated by this problem, we intend to define a connectivity of a subset X of the vertex set of a graph and prove a theorem similar to Thomas' and Yu's result if X is 4-connected in this sense.

Given a subset $\emptyset \neq X \subseteq V(G)$ of G, we define the connectivity of X in G. A set $S \subset V(G)$ is an *X-separator* of G if the graph $G - S$ obtained from G by removing S contains at least two components each containing a vertex of X. Let $\kappa_G(X)$ be the maximum integer less than or equal to $|X| - 1$ such that the cardinality of each X-separator $S \subset V(G)$ – if any exists – is at least $\kappa_G(X)$. According to this, it is clear that $\kappa_G(V(G)) = k$ if and only if G is k-connected but not $(k+1)$-connected.

For a subset $\emptyset \neq X \subseteq V(G)$ of G, Harant and Senitsch [**2**] extended Theorem 1.

Theorem 3 (Harant, Senitsch [**2**])**.** *Let G be a planar graph, $X \subseteq V(G)$, $\kappa_G(X) \geq 4$, $E \subseteq E(G[X])$, and $|E| \leq 2$. Then G has a cycle containing X and E.*

Here we prove the forthcoming Theorem 4, which extends Theorem 2.

Theorem 4. *Let G be a planar graph, $X \subseteq V(G)$, $\kappa_G(X) \geq 4$, and $M \subseteq X$ with $|M| \leq 2$. Then $G - M$ contains a cycle containing $X \setminus M$.*

In case $X = V(G)$, Theorem 2 is a consequence of Theorem 4; thus, the conditions $\kappa_G(X) \geq 4$ and $|M| \leq 2$ cannot be weakened.

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Before we start to prove our Theorem 4, we introduce the concept of bridges and Tutte paths which the proofs of Theorems 1 and 2 are principally based on. Therefore, let G be a 2-connected graph embedded into the plane, H be a subgraph of G, $V(G) \setminus V(H) \neq \emptyset$, and F be a component of $G - V(H)$. If $N_G(F) \subseteq V(H)$ is the set of neighbours of F in $V(H)$, then B with $V(B) = V(F) \cup N_G(F)$ and $E(B) = E(F) \cup \{uv \in E(G) \mid u \in V(F), v \in V(H)\}$ is a non-trivial *bridge* of H, where $N_G(F)$ and $V(F)$ are called the set $T(B)$ of *touch vertices* and the set $I(B)$ of *inner vertices* of B, respectively. It should be added that a *trivial bridge* is a subgraph of G isomorphic to K_2 with both end vertices but not its edge in H. Since we are interested in bridges containing a vertex of X, all references to bridges focus to non-trivial ones.

The *exterior cycle* of G is the cycle C_G bounding the infinite face of G. Thomas and Yu [**6**] generalized the terms of Tutte in the following sense. Let $E \subseteq E(G)$ for a graph G, then a path P of G on at least two vertices is an *E-snake* of G if each bridge of P has at most three touch vertices and each bridge containing an edge of E has two touch vertices. A *Tutte path* in its original meaning is an

$E(C_G)$-snake. A cycle C of G is an *E-sling* of G if $C - e$ for some $e \in E(C)$ is an E-snake.

Tutte [**8**] proved that, for $x, y \in V(C_G)$ and $e \in E(C_G)$, G contains a Tutte path from x to y containing e. Thomassen [**7**] improved Tutte's result by removing the restriction on the location of y, and, eventually, Sanders [**5**] established Lemma 1.

Lemma 1 (Sanders [**5**]). *If G is a 2-connected plane graph, $e \in E(C_G)$, and $x, y \in V(G)$, then G has an $E(C_G)$-snake from x to y containing e.*

Tutte's result is also generalized by Thomas and Yu.

Lemma 2 (Thomas, Yu [**6**]). *If G is a 2-connected plane graph with outer cycle C_G, another facial cycle C_2, and $e \in E(C_G)$, then G has an $(E(C_G) \cup E(C_2))$-sling C such that $e \in E(C)$ and no C-bridge contains edges of both C_G and C_2.*

Before we start the proof of Theorem 4, we prove a lemma that provides a sufficient connectivity of G.

Lemma 3. *Let G be a graph, $S \subset V(G)$ be a minimal $V(G)$-separator of G, and F be a component of $G - S$. Furthermore, let $X \subseteq V(F) \cup S$ and $G' = G(V(F), S)$ be the graph obtained from $G[V(F) \cup S]$ by adding all possible edges between vertices of S (if not already present). Then $\kappa_{G'}(X) \geq \kappa_G(X)$. Furthermore, $G' = G(V(F), S)$ is planar if G is planar and $|S| \leq 3$.*

Proof. The first part follows from Lemma 2 in [**2**]. Minimality of S ensures the planarity of $G(V(F), S)$. □

Proof of Theorem 4.
Suppose, to the contrary, that Theorem 4 does not hold and let G, X, M form a counterexample such that $|V(G)|$ is minimum. By Lemma 3, $M \neq \emptyset$. Here we consider just the case that $M = \{x_1, x_2\}$ and $x_1 \neq x_2$. The case $x_1 = x_2$, i.e. $|M| = 1$, follows with the same forthcoming arguments and is left to the reader.

If G is not 2-connected, then there will be a block F of G with $X \subseteq V(F)$ since $\kappa_G(X) \geq 4$. By Lemma 3, $\kappa_F(X) \geq 4$ and F is a smaller counterexample than G, a contradiction.

Assume that G is embedded in the plane such that x_1 is incident with the outer face and consider $G - \{x_1, x_2\}$. Since $|X| \geq 5$ (because $\kappa_G(X) \geq 4$) and $\kappa_{(G-\{x_1,x_2\})}(X \setminus \{x_1, x_2\}) \geq 2$, there is a block H of G containing $X \setminus \{x_1, x_2\}$.

Assume there is a component K of $G - (\{x_1, x_2\} \cup V(H))$ and let $N_G(K)$ be the neighbours of K in G. Because H as a block of G is a maximal 2-connected subgraph, it follows $|N_G(K) \cap V(H)| \leq 1$. Obviously, $N_G(K) \setminus V(H) \subseteq \{x_1, x_2\}$ and, therefore, $|N_G(K)| \leq 3$. Consider the graph G' obtained from G by removing $V(K)$ and adding all edges between the vertices of $N_G(K)$. Then G' is planar since $|N_G(K)| \leq 3$ and, furthermore, $\kappa_{G'}(X) \geq 4$ (see Lemma 3). By the choice of G, there is a cycle C of G' containing all vertices of X except x_1 and x_2. Evidently, C misses all new edges between the vertices of $N_G(K)$, thus, C is also a cycle of G, a contraction. We conclude that $H = G - \{x_1, x_2\}$.

For $j = 1, 2$, there are (not necessarily distinct) faces α_j of H containing the vertex x_j in G and let C_j be the facial cycle of α_j in H. Because of the choice of the embedding of G, α_1 is the outer face of H, thus, $C_H = C_1$.

We follow the proof in [**6**] and assume first that $C_1 = C_2$. If $\alpha_1 \neq \alpha_2$, then $H = C_1$ and C_1 is the desired cycle. Otherwise, the vertices of $V(C_1)$ can be numbered with $v_1, v_2, \dots, v_k$ according to their cyclic order in a such way that x_2 is not adjacent to vertices $v_2, v_3, \dots, v_{i-1}$ and x_1 is not adjacent to vertices $v_{i+1}, v_{i+2}, \dots, v_k$ for some integer i with $3 \leq i \leq k-1$; note that x_1 and x_2 have degree at least 4 in G. We apply Lemma 1 and consider an $E(C_1)$-snake Q of H from v_1 to v_2 containing v_iv_{i+1} which can be joined by v_1v_2 to a cycle. Since G is a counterexample, there is $x \in X \setminus (V(Q) \cup \{x_1, x_2\})$ and a bridge B of Q containing x as an inner vertex. If $I(B) \cap V(C_1) = \emptyset$, then $N_G(x_1) \cap V(B) \subseteq T(B)$ and $T(B)$ separates x from x_1 in G, contracting $\kappa_G(X) \geq 4$. If $v \in I(B) \cap V(C_1)$, then the edge uv, where u is a neighbour of v at C_1, belongs to B. Especially, $u \in V(B)$ and B has exactly two attachment points s and t in $V(Q)$ and $s, t \in V(C_1)$. Thus, the subpath P of C_1 from s to t containing v is a path of B. If $(I(B) \cap V(C_1)) \setminus V(P) \neq \emptyset$, then there would be another subpath P' of C_1 connecting s and t with $V(P') \subseteq V(B)$; hence, $E(C_1) = E(P) \cup E(P')$, contradicting $v_iv_{i+1} \in E(Q)$.

Furthermore, $v_1, v_i \notin I(B)$ and $(I(B) \cap V(C_1)) \cap N_G(x_j) = \emptyset$ for one $j \in \{1, 2\}$. But then $N_G(x_j) \cap V(B) \subseteq T(B)$ and $T(B) \cup \{x_{3-j}\}$ separates x from x_j, contracting $\kappa_G(X) \geq 4$.

Thus, $C_1 \neq C_2$, and by Lemma 2, there is a $(E(C_1) \cup E(C_2))$-sling C. Since G is a counterexample, there is $x \in X \setminus V(C)$ and a bridge B of C containing x as an inner vertex and not simultaneously edges from both cycles C_1 and C_2. Hence, $I(B) \cap V(C_1) = \emptyset$ or $I(B) \cap V(C_2) = \emptyset$ and – in both cases – $T(B)$ separates x from x_1 or x_2 in G, contracting $\kappa_G(X) \geq 4$.

This completes the proof of Theorem 4. ☐

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