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MAXIMAL EDGE-COLORINGS OF GRAPHS

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Abstract. For graph G of order n a maximal edge-coloring is a proper partial coloring with $\chi'(K_n)$ colors such that adding any edge to G in any color makes it improper. Meszka and Tyniec proved that for some numbers of edges it is impossible to find such a graph, and provided constructions for some other numbers of edges. However, for many values, the problem remained open. We give a complete solution of this problem for all even values of n and for odd $n \geq 37$.

1. Introduction

In combinatorics there are a lot of problems that are associated with maximality of a family of some objects. When we have a subset of the space of objects that satisfies some fixed properties we say it is *maximal* when adding to this set any new element from the space results in violeting our conditions. Various types of such spaces were investigated: one-factors of K_{2n} [**3**], two-factors of K_n [**6**], triangle-factors of K_{3n} [**9**] and latin cubes [**2**], just to name a few.

One of the best-known problems in the field is related with maximal partial latin squares [**7**]. A *partial latin square* is an array with n rows and n columns such that each of its entries is filled with a number from 1 to n or left empty and each number can appear in every column and every row at most once. For an exhaustive survey on latin squares see [**4**]. The problem of determining how many non-empty cells a maximal partial latin square can have is equivalent to the problem of determining how many edges can be in the maximal partial coloring of a bipartite graph $G = (U, V; E)$ with $n = \chi'(K_{n,n})$ colors, where $|U| = |V| = n$.

As a natural consequence of the above problem, and taking into account the fact that the theory of on-line maximal edge coloring was developed ([**1**], [**5**]), Meszka and Tyniec in [**8**] analyzed maximal partial colorings of K_n with $\chi'(K_n)$ colors. The main problem is to determine for which values of m there exists a partial proper coloring of K_n with exactly m edges colored such that coloring any other edge makes it improper. The problem has been partially solved, although for $m \in [\frac{1}{4}n^2 - \frac{3}{8}n + 1; \frac{1}{4}n^2 - 1]$ when n is an even number, for $m = \frac{1}{4}n^2 + 1$ when $n \equiv 2 \pmod 4$ and for $m \in [\frac{1}{4}n^2 - \frac{1}{4}n; \frac{1}{4}n^2 + \frac{1}{2}n - \frac{7}{4}]$ when n is odd it remained

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open. By analysis of the properties and structure of graphs that yield a maximal edge coloring, we managed to solve the problem completely for all even values of n and for big enough odd values of n (namely, $n \geq 37$).

2. Statement of the problem

The edge chromatic number of a complete graph of n vertices, where $n \geq 2$ (denoted by $\chi'(K_n)$) is equal to $n-1$, when n is an even number and n, when n is an odd number. We say that a fixed graph G of order n has *maximal edge coloring* if there exists a proper edge coloring of G with $\chi'(K_n)$ colors (which set will be denoted by C) such that attaching any edge in arbitrary color from the set C to the graph G will make the coloring improper. Clearly, not every graph has a maximal edge coloring. That is why for the fixed number $n \in \mathbb{N}, n \geq 3$ we define a *spectrum* (denoted by $MEC(n)$) as a set of all numbers of edges m for which there exists a graph of order n and size m that has a maximal edge coloring:

$$\begin{aligned} MEC(n) = \{m \in \mathbb{N} : &\text{ there exists a graph } G \text{ such that } |V(G)| = n \\ &\text{and } |E(G)| = m \text{ which has a maximal edge coloring}\}. \end{aligned}$$

We say that vertex $v \in V(G)$ *can see* some color $c \in C$ if there exists an edge colored c adjacend to v. If a coloring is edge-maximal, then any vertices u, v such that $uv \notin E$ must see togheter all colors.

A problem of determing $MEC(n)$ for $n \geq 3$ was studied by Meszka and Tyniec [**8**]. In their paper they proved the following two theorems.

Theorem 1. *Let n be an even number, $n > 10$.*
- *If $\frac{1}{4}n^2 \leq m \leq \frac{1}{2}n^2 - \frac{1}{2}n = \binom{n}{2}$, $m \neq \binom{n}{2} - 1$, and for $n \equiv 2 \pmod 4$, $m \neq \frac{1}{4}n^2 + 1$, then $m \in MEC(n)$.*
- *If $0 \leq m \leq \frac{1}{4}n^2 - \frac{3}{8}n$ or $m = \binom{n}{2} - 1$, then $m \notin MEC(n)$.*

Theorem 2. *Let n be an odd number, $n > 10$.*
- *If $\frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4} \leq m \leq \frac{1}{2}n^2 - \frac{1}{2}n = \binom{n}{2}$, then $m \in MEC(n)$.*
- *If $0 \leq m \leq \frac{1}{4}n^2 - \frac{1}{4}n - 1$, then $m \notin MEC(n)$.*

Morover, by computer analysis the spectra for $3 \leq n \leq 10$ were completely determined.

From the above theorems it is clearly seen that the problem has not been solved for $m \in [\frac{1}{4}n^2 - \frac{3}{8}n + 1, \frac{1}{4}n^2 - 1]$ when $n \geq 12$ and even, for $m = \frac{1}{4}n^2 + 1$ when $n \geq 14$, $n \equiv 2 \pmod 4$ and for $m \in [\frac{1}{4}n^2 - \frac{1}{4}n, \frac{1}{4}n^2 + \frac{1}{2}n - \frac{7}{4}]$ when $n \geq 11$ and odd.

3. Main results

We continued the investigation of maximal partial colorings of K_n and by analysis of the structure and properties of graphs that yield a maximal edge-coloring, we managed to show that for all values of m for which the problem remained open, it is not possible to construct a graph of size m that has a maximum edge-coloring. This solves the problem completely for even n and for odd n not smaller than 37.

More exactly, we proved the following three theorems.

Theorem 3. *Let n be an even number, $n \geq 4$. If $m \in MEC(n)$, then $m \geq \frac{1}{4}n^2$.*

Theorem 4. *Let n be an even number, $n \geq 14$, $n \equiv 2 \pmod 4$. Then $\frac{1}{4}n^2 + 1 \notin MEC(n)$.*

Theorem 5. *Let n be an odd number, $n \geq 37$. If $m \in MEC(n)$, then $m \geq \frac{1}{4}n^2 + \frac{1}{2}n - \frac{3}{4}$.*

To prove Theorem 3 we compute the bound for the number of edges that the sum of degrees of non-adjacent vertices implies, and consider a stucture of a graph that yields a maximal edge-coloring. To prove Theorem 4 it is essential to analyze the structure of a graph and the distribution of colors. The proof of Theorem 5 uses similar techniques but is more complicated because of bigger number of cases and more complex structures that appear.

4. Overview of the proof for even n

The proof of Theorem 3 consists of two cases which are the consequences of a natural observation: if we have a maximal edge-coloring of a graph, then every pair of non-adjacent vertices $u, v \in V$ must see toghether all $\chi'(K_n) = n-1$ colors, thus $\deg(u) + \deg(v) \geq n - 1$ for every $u, v \in V$ such that $uv \notin E$. The first case is that for all non-adjacent pairs of vertices, the sum of their degrees is at least n. By suming up these inequalities over all such pairs, it is possible to show that m must be at least $\frac{1}{4}n^2$. In the second case, there exists a not connected pair of vertices $u, v \in V$ with sum of degrees equal to $n - 1$. If A is the set of colors that u can see, then v must see the set $\bar{A}$ of all the remaining colors. It can be also proven that u and v must have exactly one common neighbor w and the set of vertices $V \setminus \{u, v, w\}$ can be split into two subsets: the set X of vertices connected with u and the set Y of vertices connected with v. This situation is presented on Figure 1. If we use information about this structure, we can estimate from below the sum of degrees of all vertices, which gives the desired number of edges.

To prove Theorem 4, we notice that if a graph has a maximal edge-coloring and number of edges equal to $\frac{1}{4}n^2+1$, then it has very specific structure and distrubion of colors. Such a structure and discribution do not exist as we need an additional edge to keep the coloring maximal.

5. Overview of the proofs for odd n

For odd n there are $\chi'(K_n) = n$ colors, thus for any two non-adjacent vertices u, v we have $\deg(u)+\deg(v) \geq n$. This time we need to consider three cases. In the first case we assume that $\deg(u) + \deg(v) \geq n + 2$ for any pair of non-adjacent vertices and we sum up the inequalities over all pairs of non-adjacent vertices to obtain the desired number of edges. The second case is that there exists a non-adjacent pair of vertices u, v such that $\deg(u) + \deg(v) = n$. We obtain a similar structure which was presented in the overview of the proof for even n. The differences are

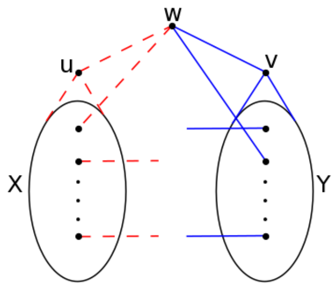

Figure 1. The structure of a graph of even order n that yields a maximal coloring when there are two vertices of sum of degrees equal to $n-1$. Dashed red edges indicate colors from the set A, solid blue edges – colors from the set $\bar{A}$.

that now u and v have exactly two common neighbors and to obtain appropriate number of edges, we aditionally need to use the fact that one of the sets: $X \cup \{u\}$ or $Y \cup \{v\}$ forms a clique of even order with edges in colors only from A or from $\bar{A}$, and also that there exists a perfect matching in one of the colors.

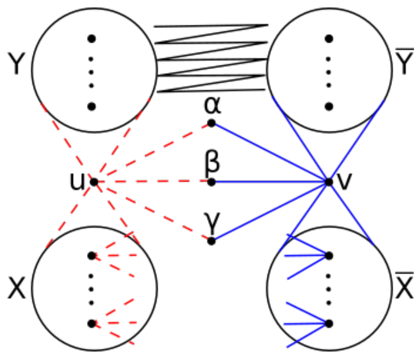

Figure 2. The graph of odd order n in the third case of the proof of Theorem 5. Dashed red edges indicate colors from the set $A \cup \{c\}$, the solid blue edges – colors from the set $\bar{A} \cup \{c\}$.

In the third case we assume that any pair of non-adjacent vertices has sum of degrees at least $n+1$ and there exists a non-adjacent pair $u, v \in V$ such that $\deg(u) + \deg(v) = n+1$. Then u and v can see exactly one common color c, so u can see colors from $A \cup \{c\}$ and v can see colors from $\bar{A} \cup \{c\}$, where $\bar{A}$ is the complement of $A \cup \{c\}$. They must have also exactly three common neighbors: α, β and γ. All vertices from $V \setminus \{u, v, \alpha, \beta, \gamma\}$ are connected with one of the vertices u and v and we can write this set as a disjoint union of four sets: $X, Y, \bar{X}, \bar{Y}$, where $X \cup Y$ is the set of vertices connected with u, but the elements of X can see color c, while the elements of Y cannot see it (in analogical way we define sets $\bar{X}$ and $\bar{Y}$ of vertices connected with v). Aditionally, each pair of vertices $y \in Y$ and $\bar{y} \in \bar{Y}$ must be connected. This situation is presented on the Figure 2. We have to consider four cases depending on the number of elements of smaller of the sets Y and $\bar{Y}$ to complete the argument.

In this last case of the proof for odd n, estimations that we used require to have $n \geq 37$. Smaller values of n would require longer and more careful analysis of significantly more cases, which would make the paper extremaly technical. Taking into consideration the fact that we have a lot of information about the structure of graphs that yield maximal edge-coloring, it is possible to determine the spectra for these values by exhaustive computer analysis of the described cases. Thus we decided to omit those smaller values of n.

References

1. Bar-Noy A., Motwani R. and Naor J., *The greedy algorithm is optimal for on-line edge coloring*, Inform. Process. Lett. **44** (1992), 251–253.

2. Britz T., Cavenagh N. J. and Sørensen H. K., *Maximal partial Latin cubes*, Electron. J. Combin. **22** (2015), #P1.81.

3. Cousins E. A. and Wallis W. D., *Maximal sets of one-factors* in: Combinatorial Mathematics III, Springer, Berlin, 1974, 90–94.

4. Dénes J. and Keedwell D., *Latin Squares and their Applications*, North-Holland, Amsterdam, 2015, 83–122.

5. Favrholdt L. M. and Nielsen M. N., *On-line edge-coloring with a fixed number of colors*, Algorithmica **35** (2003), 176–191.

6. Hoffman D. G., Rodger C. A. and Rosa A., textitMaximal sets of 2-factors and Hamiltonian cycles, J. Combin. Theory Ser. B **57** (1993), 69–75.

7. Horák P. and Rosa A., *Maximal partial latin squares* in: Graphs, Matrices and Designs (R. Rees, eds.), Kluwer, 1993, 225–233.

8. Meszka M. and Tyniec M., *Maximal edge-colorings of graphs*, Graphs Combin. **33** (2017), 1451–1458.

9. Rees R. S., Rosa A. and Wallis W. D., *Maximal sets of triangle-factors*, Australas. J. Combin. **9** (1994), 67–108.

10. Rosa A., *Maximal partial designs and configurations*, Matematiche (Catania) **45** (1990), 149–162.

11. Rosa A., *Maximal designs and configurations – a survey*, Acta Univ. M. Belii Ser. Math. **23** (2015), 9–25.

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