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COHOMOLOGY GROUPS OF NON-UNIFORM RANDOM SIMPLICIAL COMPLEXES

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ABSTRACT. We consider a model of a random simplicial complex generated by taking the downward-closure of a non-uniform binomial random hypergraph, in which each set of $k+1$ vertices forms an edge with some probability p_k independently, where p_k depends on k and on the number of vertices n. We consider a notion of connectedness on this model according to the vanishing of cohomology groups over an arbitrary abelian group R. We prove that this notion of connectedness displays a phase transition and determine the threshold. We also prove a hitting time result for a natural process interpretation, in which simplices and their downward-closure are added one by one.

1. INTRODUCTION

1.1. Motivation

One of the first and most famous results in the theory of random graphs, due to Erdős and Rényi [**5**], states that the uniform random graph $G(n,m)$ displays a phase transition threshold for the property of being connected at about $m = \frac{1}{2}n\log n$. Almost equivalenty, in modern terminology, the binomial random graph $G(n,p)$ becomes connected around $p = \frac{\log n}{n}$. The result was subsequently strengthened by Bollobás and Thomason [**1**] to a *hitting time* result – the random graph process, in which edges are added to an empty graph one by one in a uniformly random order, is very likely to become connected at exactly the moment at which the last isolated vertex disappears (i.e. acquires an edge), which occurs around time $p = \frac{\log n}{n}$.

There have been many generalisations of these results to higher-dimensional analogues of graphs, including hypergraphs (e.g. [**3, 7**]) and simplicial complexes (e.g. [**2, 7, 8, 9**])

In this paper we prove a generalisation for a model of simplicial complexes arising from non-uniform binomial random hypergraphs. The notion of connectedness that we study regards the vanishing of cohomology groups over an abelian group.

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This extends recent results of [**2**] in two important ways: firstly in that the random hypergraph used to generate the random complex is *non-uniform*, having different probabilities for each edge size; and secondly in that we consider cohomology groups over *any* abelian group R, rather than just over $\mathbb{F}_2$. It is also closely related to results of Linial and Meshulam [**8**], Meshulam and Wallach [**9**], and Kahle and Pittel [**7**], although the model of random simplicial complexes that we consider is different and leads to significantly more complex behaviour.

1.2. Model

Throughout the paper let $d \geq 2$ be a fixed integer and let R be an abelian group with at least two elements. We use additive notation for the group operation of R and denote its neutral element by 0_R. Given an integer k, we write $[k] := \{1, \ldots, k\}$.

We define a model of a random d-complex generated from a non-uniform random hypergraph, in which sets of vertices have different probabilities of forming an edge depending on their size.

Definition 1.1. For each $k \in [d]$, let $p_k = p_k(n) \in [0,1] \subset \mathbb{R}$ be given and write $\mathbf{p} := (p_1, \ldots, p_d)$. Denote by $G_{\mathbf{p}} = G(n, \mathbf{p})$ the (non-uniform) binomial random hypergraph on vertex set $[n]$ in which, for all $k \in [d]$, each element of $\binom{[n]}{k+1}$ forms a hyperedge with probability p_k independently. By $\mathcal{G}_{\mathbf{p}} = \mathcal{G}(n, \mathbf{p})$, we denote the random d-dimensional simplicial complex on $[n]$ such that

- the 0-simplices of $\mathcal{G}_{\mathbf{p}}$ are the singletons of $[n]$ and
- for each $i \in [d]$, the i-simplices are precisely the $(i+1)$-sets which are contained in hyperedges of $G_{\mathbf{p}}$.

In other words, $\mathcal{G}_{\mathbf{p}}$ is the downward-closure of the set of hyperedges of $G_{\mathbf{p}}$, together with all singletons of $[n]$ (if those are not already in the downward-closure).

Recently, a similar but slightly more general model was independently introduced in [**6**]. This model has a similar flavour to the multi-parameter model introduced by Costa and Farber [**4**].

Denote by $H^i(\mathcal{G}; R)$ the i-th cohomology group of a simplicial complex $\mathcal{G}$ with coefficients in R. It is well-known that $H^0(\mathcal{G}; R) = R$ if and only if $\mathcal{G}$ is connected in the topological sense (see e.g. [**10**, Theorem 42.1]), which we call *topologically connected* in order to distinguish it from other notions of connectedness. For any positive integer i, the *vanishing* of $H^i(\mathcal{G}; R)$ can be viewed as a "higher-order connectedness" of $\mathcal{G}$.

Definition 1.2. Given a positive integer j, a simplicial complex $\mathcal{G}$ is called *R-cohomologically j-connected* (*j-cohom-connected* for short) if

(i) $H^0(\mathcal{G}; R) = R$;
(ii) $H^i(\mathcal{G}; R) = 0$ for all $i \in [j]$.

1.3. Main results

The main aim of this paper is to provide an analogue of the graph results of Erdős and Rényi [**5**] and of Bollobás and Thomason [**1**] for the $\mathcal{G}_{\mathbf{p}}$ model of random simplicial complexes. Given an appropriate direction $\bar{\mathbf{p}}$, we consider the random simplicial complex process $(\mathcal{G}_\tau) = (\mathcal{G}_{\tau\bar{\mathbf{p}}})_{\tau\in\mathbb{R}_{\geq 0}}$ (defined more formally later) and describe for which values of τ the complex is j-cohom-connected and for which it is not, relating this threshold to the disappearance of the last minimal obstruction.

In order to define the minimal obstructions $\hat{M}_j^k$, we introduce the following necessary concepts.

Definition 1.3. Let $j \in [d-1]$ and let k be an integer with $j+1 \leq k \leq d$. Given a k-simplex K in a d-dimensional simplicial complex $\mathcal{G}$, we say that a collection $\mathcal{F} = \{P_0, \dots, P_{k-j}\}$ of j-simplices forms a *j-flower in K* (see Figure 1) if

(F1) $K = \bigcup_{i=0}^{k-j} P_i$;
(F2) $C := \bigcap_{i=0}^{k-j} P_i$ satisfies $|C| = j$.

We call the j-simplices P_i the *petals* and the set C the *centre* of the j-flower $\mathcal{F}$.

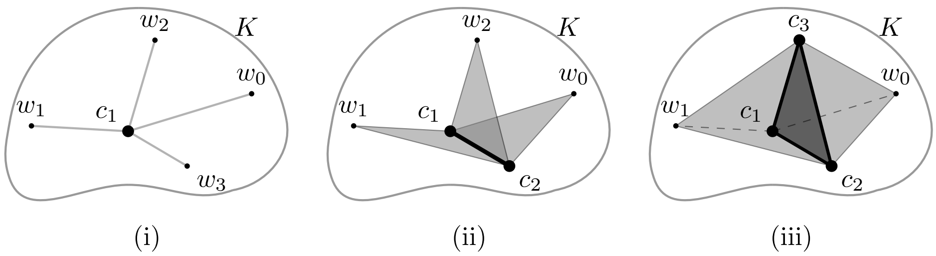

Figure 1. Examples of j-flowers in a k-simplex K, for $k = 4$ and $j = 1, 2, 3$.

(i) The 1-flower in K with centre $C = \{c_1\}$ (bold black) and petals $P_i = C \cup \{w_i\}$, $i = 0, 1, 2, 3$ (grey).
(ii) The 2-flower in K with centre $C = \{c_1, c_2\}$ (bold black) and petals $P_i = C \cup \{w_i\}$, $i = 0, 1, 2$ (grey).
(iii) The 3-flower in K with centre $C = \{c_1, c_2, c_3\}$ (bold black) and petals $P_i = C \cup \{w_i\}$, $i = 0, 1$ (grey).

Observe that for each k-simplex K and each $(j-1)$-simplex $C \subseteq K$, there is a unique j-flower in K with centre C, namely

$$\mathcal{F}(K, C) := \{C \cup \{w\} \mid w \in K \setminus C\}. \tag{1}$$

Definition 1.4. For any $(j+2)$-set A in a complex $\mathcal{G}$, the collection of all $(j+1)$-sets of A is called a *j-shell* if each of them forms a j-simplex in $\mathcal{G}$.

If the collection of all $(j+1)$-subsets of a $(j+2)$-set A forms a j-shell, with a slight abuse of terminology we also refer to the set A itself as a j-shell.

Definition 1.5. For $j+1 \leq k \leq d$, a 4-tuple (K, C, w, a) is said to form a *copy of $\hat{M}_j^k$* if

(M1) K is a k-simplex in $\mathcal{G}$;
(M2) C is a $(j-1)$-simplex in K such that every simplex of $\mathcal{G}$ that contains a petal of the flower $\mathcal{F} = \mathcal{F}(K, C)$ is contained in K;

(M3) $w \in K \setminus C$ and $a \in [n] \setminus K$ are such that $C \cup \{w\} \cup \{a\}$ is a j-shell in $\mathcal{G}$.

We call the j-simplex $C \cup \{w\}$ the *base* and a the *apex vertex* of the j-shell $C \cup \{w\} \cup \{a\}$. Every other j-simplex in $C \cup \{w\} \cup \{a\}$ is called a *side* of the j-shell.

A copy of $\hat{M}_j^k$ is an obstruction to the vanishing of $H^j(\mathcal{G}; R)$, since it is easy to define a *bad function*, i.e. a j-cocycle which is not a j-coboundary, by choosing any non-zero element $r \in R \setminus \{0_R\}$ and assigning the values $\pm r$ to the (ordered) petals in an appropriate way. Furthermore, it can be shown that a copy of $\hat{M}_j^k$ is a *minimal* obstruction in terms of the size of the support of a bad function.

The random d-complex $\mathcal{G}_{\mathbf{p}}$ can be turned into a *process*, by assigning a *birth time* to each k-simplex. More precisely, for each $k \in [d]$ and each $(k+1)$-set $K \in \binom{[n]}{k+1}$ independently, sample a birth time uniformly at random from $[0,1]$. Then $\mathcal{G}_{\mathbf{p}}$ is exactly the complex generated by the $(k+1)$-sets with birth times at most p_k, for all $k \in [d]$, by taking the downward-closure. If we fix a "direction" $\bar{\mathbf{p}} = (\bar{p}_1, \dots, \bar{p}_d)$ of non-negative real numbers with $\bar{p}_d \neq 0$, set

$$\mathbf{p} = \tau\bar{\mathbf{p}} := (\min\{\tau p_1, 1\}, \dots, \min\{\tau p_d, 1\}),$$

and gradually increase τ from 0 to $\tau_{\max} := 1/\bar{p}_d$, then $\mathcal{G}_{\mathbf{p}}$ becomes a process in which simplices (together with their downward-closure) arrive one by one. We will denote this process by $(\mathcal{G}_{\tau\bar{\mathbf{p}}})_\tau$, or sometimes just by $(\mathcal{G}_\tau)$ when the direction $\bar{\mathbf{p}}$ is clear from the context.

Given a direction $\bar{\mathbf{p}}$, and a $(k+1)$-set K with birth time t_K, the *scaled birth time* of K is

$$\tau_K := \frac{t_K}{\bar{p}_k}.$$

Thus τ_K is distributed uniformly in $[0, \bar{p}_k^{-1}]$, and $\mathcal{G}_\tau = \mathcal{G}_{\tau\bar{\mathbf{p}}}$ consists of all those simplices with *scaled* birth time at most τ, together with their downward-closure.

With some elementary arguments we can show that, when considering j-cohom-connectedness, it suffices to consider a direction $\bar{\mathbf{p}}$ with some specific properties, namely that for each $1 \leq k \leq d$ there are constants $\bar{\alpha}_k, \bar{\gamma}_k$ and a function $\bar{\beta}_k = \bar{\beta}_k(n)$ such that

$$\bar{p}_k = \frac{\bar{\alpha}_k \log n + \bar{\beta}_k}{n^{k-j+\bar{\gamma}_k}}(k-j)!, \tag{2}$$

and furthermore

- (P1) at least one of $\bar{\alpha}_k, \bar{\gamma}_k$ is zero and neither of them is negative;
- (P2) if $\bar{\alpha}_k = 0$, then $\bar{\beta}_k$ is either zero or it is positive and *subpolynomial* in the sense that for every constant $\varepsilon > 0$, we have $\bar{\beta}_k = o(n^\varepsilon)$, but $\bar{\beta}_k = \omega(n^{-\varepsilon})$;
- (P3) if $\bar{\gamma}_k = 0$, then $|\bar{\beta}_k| = o(\log n)$;
- (P4) there exists an index $j+1 \leq k_0 \leq d$ with $\bar{\alpha}_{k_0} > 0$.

Additionally, for all indices k with $j \le k \le d$ and $\bar{p}_k \neq 0$, if we define the parameters

$$
\begin{aligned}
\bar{\lambda}_k &:= j+1-\bar{\gamma}_k-(k-j+1)\sum_{i=j+1}^{d}\bar{\alpha}_i,\\
\bar{\mu}_k &:= -(k-j+1)\sum_{i=j+1}^{d}\frac{\bar{\beta}_i}{n^{\bar{\gamma}_i}}+\begin{cases}\log\log n & \text{if } \bar{\alpha}_k\neq 0,\\ \log(\bar{\beta}_k) & \text{if } \bar{\alpha}_k=0,\end{cases}\\
\bar{\nu}_k &:= \begin{cases}-\log((j+1)!) & \text{if } k=j,\\ -\log(j!)-\log(k-j+1)+\log(\bar{\alpha}_k) & \text{if } \bar{\alpha}_k\neq 0,\\ -\log(j!)-\log(k-j+1) & \text{otherwise,}\end{cases}
\end{aligned}
\tag{3}
$$

then we may assume that

(C1) $\bar{\lambda}_k \log n + \bar{\mu}_k + \bar{\nu}_k \le 0$, for all indices k with $j \le k \le d$ and $\bar{p}_k \neq 0$,
(C2) $\bar{\lambda}_{\bar{k}} \log n + \bar{\mu}_{\bar{k}} + \bar{\nu}_{\bar{k}} = 0$, for some $\bar{k}$ with $j \le \bar{k} \le d$.

The definitions of the parameters $\bar{\lambda}_k, \bar{\mu}_k, \bar{\nu}_k$ are motivated by Lemma 2.2. We call a direction $\bar{\mathbf{p}}$ satisfying (P1)–(P4) and (C1)–(C2) a *j-critical direction.*

Theorem 1.6. *For $j \in [d-1]$ and a j-critical direction $\bar{\mathbf{p}}$, let $\mathbf{p}^* = \tau^*\bar{\mathbf{p}}$, where*

$$\tau^* := \sup\{\tau \in \mathbb{R}_{\ge 0} \mid \mathcal{G}_{\tau\bar{\mathbf{p}}} \text{ contains a copy of } \hat{M}_j^k \text{ for some } j \le k \le d\}.$$

Then for every function ω of n which tends to infinity as $n \to \infty$, the following statements hold with high probability.

(i) $\tau^* = 1 + o\left(\frac{\omega}{\log n}\right)$.
(ii) *The random d-complex process $(\mathcal{G}_\tau) = (\mathcal{G}_{\tau\bar{\mathbf{p}}})_\tau$ is not R-cohomologically j-connected for all $\tau < \tau^*$, i.e.*

$$H^0(\mathcal{G}_{\mathbf{p}}; R) \neq R \quad \text{or} \quad H^i(\mathcal{G}_{\mathbf{p}}; R) \neq 0 \text{ for some } i \in [j],$$

for $\mathbf{p} = \tau\bar{\mathbf{p}}$ and for $\tau \in [0, \tau^*)$.
(iii) *The random d-complex process $(\mathcal{G}_\tau) = (\mathcal{G}_{\tau\bar{\mathbf{p}}})_\tau$ is R-cohomologically j-connected for all $\tau \ge \tau^*$, i.e.*

$$H^0(\mathcal{G}_{\mathbf{p}}; R) = R \quad \text{and} \quad H^i(\mathcal{G}_{\mathbf{p}}; R) = 0 \text{ for all } i \in [j],$$

for $\mathbf{p} = \tau\bar{\mathbf{p}}$ and for $\tau \in [\tau^*, \tau_{\max}]$.

Let us note that neither j-cohom-connectedness nor the presence of copies of $\hat{M}_j^k$ are necessarily monotone properties, which makes the proofs significantly harder. Indeed, it is not immediately clear that j-cohom-connectedness should have a single threshold – in principle, $\mathcal{G}_\tau = \mathcal{G}_{\tau\bar{\mathbf{p}}}$ could switch between being j-cohom-connected or not several times. However, Theorem 1.6 implies that with high probability it does not.

2. Minimal obstructions and the hitting time

Let us now define a 'reduced' version of $\hat{M}_j^k$, denoted by M_j^k, by omitting the condition (M3) on the j-shell $C \cup \{w\} \cup \{a\}$ in Definition 1.5. It is easy to see that this shell is very likely to exist if τ is 'large enough' ($\tau \geq \varepsilon/(\log n)$ will do), which will be the case well before the critical range for the disappearance of M_j^k. Thus it is convenient to switch attention from $\hat{M}_j^k$ to M_j^k.

Definition 2.1. Let k be an integer with $j+1 \leq k \leq d$. A pair (K, C) is called a *copy of* M_j^k if it satisfies the following conditions.

(M1) K is a k-simplex in $\mathcal{G}$;
(M2) C is a $(j-1)$-simplex in K such that every simplex of $\mathcal{G}$ that contains a petal of the flower $\mathcal{F} = \mathcal{F}(K, C)$ is contained in K.

For $k = j$, an isolated j-simplex is called a copy of M_j^j and of $\hat{M}_j^j$.

Lemma 2.2. *Suppose* $\tau = O(1)$ *and* $\bar{\mathbf{p}}$ *is a direction vector satisfying* (P1)–(P4)*. Then the number* X_j^k *of copies of* M_j^k *in* $\mathcal{G}_\tau$ *satisfies*

$$\log \mathbb{E}(X_j^k) = \lambda_k \log n + \mu_k + \nu_k + o(1) \tag{4}$$

for all $j \leq k \leq d$ *with* $p_k \neq 0$*, where* λ_k, μ_k, ν_k *are defined analogously to* (3) *for* $\mathbf{p} = \tau\bar{\mathbf{p}}$ *instead of* $\bar{\mathbf{p}}$.

Since $\bar{\mathbf{p}}$ in Theorem 1.6 was chosen to be j-critical, i.e. such that the right hand side of (4) is at most $o(1)$ for all k and for $\mathbf{p} = \bar{\mathbf{p}}$, this tells us that heuristically, the last minimal obstruction should disappear around time $\tau = 1$.

Indeed, let τ' denote the smallest scaled birth time $\tau \geq 1 - \frac{\log\log n}{10d \log n}$ such that $\mathcal{G}_\tau$ is M_j^k-free for all k. Extending Lemma 2.2 with a second moment argument shows that τ' is close to 1 with high probability. Furthermore, we can also show that new copies of M_j^k are unlikely to appear after time $1 - o(1)$. Together with the fact that the existence of M_j^k and $\hat{M}_j^k$ are essentially equivalent events, we obtain the following corollary.

Corollary 2.3. *Let* $\omega = \omega(n)$ *be a function that tends to infinity as* $n \to \infty$*. Then with high probability*

$$1 - \frac{\omega}{\log n} < \tau^* = \tau' < 1 + \frac{\omega}{\log n}.$$

3. Covering the subcritical case

To prove statement (ii) of Theorem 1.6, we first observe that initially the process is not topologically connected.

Lemma 3.1. *There exist positive constants* $c^- = c^-(d)$ *and* $c^+ = c^+(d)$ *such that*

(i) *Whp* $\mathcal{G}_\mathbf{p}$ *is not topologically connected if* $p_i \leq \frac{c^- \log n}{n^i}$ *for all* $i \in [d]$;
(ii) *Whp* $\mathcal{G}_\mathbf{p}$ *is topologically connected if* $p_i \geq \frac{c^+ \log n}{n^i}$ *for some* $i \in [d]$.

The proof is an elementary extension of the graph case: for (i) we use a second moment method to show that with high probability there are isolated vertices. Statement (ii) can be proved analogously to the graph case, or follows as a special case of far stronger results in [**3**] or [**11**].

Subsequently, we proceed by induction on j, and the induction hypothesis tells us that up to time $\tau = \frac{\Theta(1)}{n}$, whp the process is not even $(j-1)$-cohom-connected. (Lemma 3.1 effectively provides the base case $j = 0$ of the induction.) It remains to cover the range $\tau \in [\varepsilon/n, \tau^*)$ for some sufficiently small constant ε. We achieve this by splitting the range into three subintervals.

Lemma 3.2. *For every constant $\varepsilon > 0$ there exists a constant $\delta > 0$ such that the following is true. Let*

$$I_1 := \left[\frac{\varepsilon}{n}, \frac{\delta}{n}\right], \qquad I_2 := \left[\frac{\delta}{n}, 1 - \frac{1}{(\log n)^{1/3}}\right], \qquad I_3 := \left[1 - \frac{1}{(\log n)^{1/3}}, \tau^*\right).$$

Then with high probability there exist k_1, k_2, k_3 and for each $i = 1, 2, 3$ a copy of $\hat{M}_j^{k_i}$ that is present throughout the range I_i.

The proof strategy is to show, via a second moment argument, that there are many copies of $\hat{M}_j^k$ for some k at times $\tau = \varepsilon/n$ and $\tau = 1 - \frac{1}{(\log n)^{1/3}}$, and therefore with high probability at least one of these survives until the end of I_1 or was born by the start of I_2 respectively. On the other hand, since $\tau^* = 1 + o(1)$ whp, the last copy of $\hat{M}_j^k$ to disappear at time τ^* was already present at any time $\tau = 1 - o(1)$ whp, and therefore whp exists throughout the range I_3.

4. Supercritical case

To prove statement (iii) of Theorem 1.6, we need to show two things: firstly, that whp $\mathcal{G}_\tau$ is j-cohom-connected at time $\tau = \tau^*$, and secondly that whp it does not become disconnected later.

In order for $\mathcal{G}_\tau$ not to be j-cohom-connected, it would have to admit a j-cocycle f_τ which is not a coboundary: we call such a function f_τ a *bad function*, and denote its support by $\mathcal{S}_\tau$. Indeed, we may assume that $\mathcal{S}_\tau$ is the smallest possible support among all such bad functions in $\mathcal{G}_\tau$.

The next lemma immediately implies that for any single choice of $\tau \geq \tau^*$, whp $\mathcal{G}_\tau$ is j-cohom-connected.

Lemma 4.1. *Suppose that $\tau \geq \tau^*$.*

(i) *Let $h \in \mathbb{R}$ be any large constant. Then with high probability $|\mathcal{S}_\tau| > h$, if it exists.*
(ii) *There exists a constant $\bar{h} \in \mathbb{R}$ such that with high probability $|\mathcal{S}_\tau| < \bar{h}$, if it exists.*

Indeed, we even prove a little more than this, showing that if $\tau^* > \tau = 1 - o(1)$, then the only obstructions to j-cohom-connectedness are copies of $\hat{M}_j^k$.

The proof of this lemma exploits a very useful property of such a minimal support $\mathcal{S}_\tau$ which we call *traversability*, and which in particular implies that $\mathcal{S}_\tau$

can be explored via a search process. This allows us to bound the number of possible configurations for $\mathcal{S}_\tau$. In case (ii) we also use a result from [**9**] to give a lower bound on the number of simplices which must *not* be present in order for $\mathcal{S}_\tau$ to form the support of a j-cocycle.

Finally, to prove that whp the process $(\mathcal{G}_\tau)$ never becomes disconnected again at any time $\tau > \tau^*$, we first prove that in order for a bad function f_τ to appear with the birth of a simplex K, its support $\mathcal{S}_\tau$ would have to be K-*localised* in the sense that within $\mathcal{G}_{\tau^*}$ any simplex containing an element of $\mathcal{S}_\tau$ must lie entirely within K. We then show that with high probability in $\mathcal{G}_{\tau^*}$ there are not many candidate positions for where such a simplex K might be born, and the probability that any one of these is born before any of the simplices that would prevent $\mathcal{S}_\tau$ from being K-localised is very small, so whp no new bad function will appear.

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