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INDEPENDENT TRANSVERSALS VERSUS TRANSVERSALS

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ABSTRACT. We compare the minimum size of a vertex cover, feedback vertex set and odd cycle transversal of a graph with the minimum size of the corresponding variants in which the transversal must be an independent set. We investigate for which graphs H the two sizes are equal whenever the graph in question belongs to the class of H-free graphs. We find complete classifications for vertex cover and almost complete classifications for feedback vertex set and odd cycle transversal.

1. Introduction

A transversal $\tau(\pi)$ of a graph G is a set of vertices that transverse (intersect) all subsets of G that have some specific property π. The default aim is to find a transversal $\tau(\pi)$ that has minimum size, but one may also add further conditions, such as demanding that the transversal must induce a connected subgraph or must be an *independent set* (a set of pairwise non-adjacent vertices). In this paper we focus on the latter property and consider three classical and well-studied transversals obtained by specifying π.

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a *vertex cover* if for every edge $uv \in E$, at least one of u and v is in S, or, equivalently, if the graph $G - S$ (obtained from G by deleting the vertices in S) contains no edges. We let $\mathrm{vc}(G)$ denote the size of a minimum vertex cover of G. A set $S \subseteq V$ is a *feedback vertex set* if for every cycle in G, at least one vertex of the cycle is in S, or, equivalently, if the graph $G - S$ is a forest. We let $\mathrm{fvs}(G)$ denote the size of a minimum feedback vertex set of G. A cycle is *odd* if it has an odd number of vertices. A graph is *bipartite* if its vertex set can be partitioned into at most two independent sets. A set $S \subseteq V$ is an *odd cycle transversal* if for every odd cycle in G, at least one vertex of the cycle is in S, or, equivalently, if $G - S$ is bipartite. We let $\mathrm{oct}(G)$ denote the size of a minimum odd cycle transversal of G.

Note that under the additional constraint of being an independent set it might be possible that no such transversal exists. However, for the above three transversals it is straightforward to characterize the graphs that have an independent transversal. A graph has an independent vertex cover if and only if it is bipartite. For a bipartite graph G, we let $\mathrm{ivc}(G)$ denote the size of a minimum independent

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vertex cover. A graph has an independent feedback vertex set if and only if its vertex set can be partitioned into an independent set and a set of vertices that induces a forest. Graphs with this property are said to be *near-bipartite*. For a near-bipartite graph G, we let $\mathrm{ifvs}(G)$ denote the size of a minimum independent feedback vertex set. A graph has an independent odd cycle transversal if and only if its vertex set can be partitioned into at most three independent sets. Graphs with this property are said to be 3-*colourable*. For a 3-colourable graph G, we let $\mathrm{ioct}(G)$ denote the size of a minimum independent odd cycle transversal.

As part of a systematic study, we consider *monogenic* graph classes, which are classes of graphs defined by a single forbidden induced subgraph H. Such graphs are also said to be H-*free* (more generally, a graph is $(H_1, \ldots, H_p)$-*free* if it has no induced subgraph isomorphic to H_i for all $i \in \{1, \ldots, p\}$). Monogenic graph classes fall under the framework of hereditary graph classes, that is, classes that are closed under vertex deletion. In a previous paper **[11]** we considered the following transversal problem for all fixed graphs H: is it true that for every H-free graph G, the size of a smallest possible independent transversal (if one exists) is bounded in terms of the minimum size of a transversal? We could also formulate this by asking: is the *price of independence* bounded? As in this paper, in **[11]** we also considered the transversals vertex cover, feedback vertex set and odd cycle transversal.

We can formally define the notion of boundedness as follows. Given a class $\mathcal{X}$ of bipartite graphs, we say that $\mathcal{X}$ is ivc-*bounded* if there is a function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\mathrm{ivc}(G) \leq f(\mathrm{vc}(G))$ for every $G \in \mathcal{X}$. We define the notions of fvs-*bounded* (for near-bipartite graphs) and oct-*bounded* (for 3-colourable graphs) analogously. We will use the results of **[11]** on boundedness for H-free graphs in this paper, but to present these results we first need to introduce some additional terminology.

Let C_n, P_n and K_n denote the cycle, path and complete graph on n vertices, respectively. For $r \geq 0$, let $K_{1,r}$ denote the star on $r+1$ vertices (so $K_{1,0} = P_1$). For $r \geq 1$, let $K_{1,r}^+$ denote the graph obtained from $K_{1,r}$ by subdividing one edge. The *disjoint union* $G+H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of r copies of a graph G by rG. We now present the three results of our previous paper.

Theorem 1.1 ([**11**]). *Let H be a graph. The class of H-free bipartite graphs is* ivc-*bounded if and only if H is an induced subgraph of $K_{1,r} + rP_1$ or $K_{1,r}^+$ for some $r \geq 1$.*

Theorem 1.2 ([**11**]). *Let H be a graph. The class of H-free near-bipartite graphs is* ifvs-*bounded if and only if H is isomorphic to $P_1 + P_2$, a star or an edgeless graph.*

Theorem 1.3 ([**11**]). *Let H be a graph. The class of H-free* 3-*colourable graphs is* ioct-*bounded:*

- *if H is an induced subgraph of P_4 or $K_{1,3} + sP_1$ for some $s \geq 0$ and*
- *only if H is an induced subgraph of $K_{1,4}^+$ or $K_{1,4} + sP_1$ for some $s \geq 0$.*

We note that the classification given in Theorem 1.3 is almost complete. In [**11**] we proved that there are exactly three non-equivalent open cases. Namely, we do not know if the class of H-free 3-colourable graphs is ioct-bounded when $H = K_{1,4}$ (or equivalently $K_{1,4} + sP_1$ for any $s \geq 1$), $H = K_{1,3}^+$, or $H = K_{1,4}^+$.

Our New Results. We concentrate on determining those monogenic graph classes for which the price of independence is zero. Formally, given a class $\mathcal{X}$ of bipartite graphs, we say that $\mathcal{X}$ is ivc-*identical* if $\mathrm{ivc}(G) = \mathrm{vc}(G)$ for every $G \in \mathcal{X}$. We define the notions of ifvs-*identical* (for near-bipartite graphs) and ioct-*identical* (for 3-colourable graphs) analogously. We prove the following three classifications, the first one of which is complete.

Theorem 1.4. *Let H be a graph. The class of H-free bipartite graphs is* ivc-*identical if and only if H is an induced subgraph of $K_{1,3}^+$ or $2P_1 + P_3$.*

Theorem 1.5. *Let $H \neq K_{1,3}$ be a graph. The class of H-free near-bipartite graphs is* ifvs-*identical if and only if H is a subgraph of P_3.*

Theorem 1.6. *Let $H \notin \{K_{1,3}, K_{1,3}^+, 2P_1 + P_3\}$ be a graph. The class of H-free 3-colourable graphs is* ioct-*identical if and only if H is a subgraph of P_4 that is not isomorphic to $2P_2$.*

Related Work. The term price of independence was first used by Camby [**4**] for dominating sets (see also [**7**]). Work on the price of independence is closely related to the study of the *price of connectivity*, in which the relationship between minimum size transversals and minimum size connected transversals is studied. Starting with the work of Cardinal and Levy [**9**], this study has yielded a large number of results for a variety of transversals and other graph properties, such as vertex cover [**5, 6, 15**], dominating set [**5, 8**], face hitting set [**12**], feedback vertex set [**2**] and more general transversals [**13**]. In particular, several of these previous papers [**2, 10, 13**] also considered classes of H-free graphs.

2. Proofs

Due to space restrictions, we omit the proof of Theorem 1.6.

A *matching* in a graph G is a set of edges that have no common vertices. A graph is an *almost complete bipartite graph* if it can be obtained from a complete bipartite graph by removing a (possibly empty) set of edges that form a matching. For the proof of Theorem 1.4 we need the following lemma due to Alekseev.

Lemma 2.1 ([**1**]). *Every connected $K_{1,3}^+$-free bipartite graph is either a path, a cycle or an almost complete bipartite graph.*

We also need the following lemma for the proof of Theorem 1.4.

Lemma 2.2. *Let G be an almost complete bipartite graph. Then $\mathrm{ivc}(G) = \mathrm{vc}(G)$.*

Proof. Notice that $\mathrm{ivc}(G) = \mathrm{vc}(G)$ holds if and only if the equality holds for every connected component of G. Therefore, without loss of generality, we may assume that G is connected. Let X, Y be the parts of the bipartition of G, and let S be a minimum vertex cover of G. We may assume without loss of generality that $|X| \leq |Y|$. If $\mathrm{vc}(G) \leq 1$, then $\mathrm{ivc}(G) = \mathrm{vc}(G)$. Therefore we may assume that $|X| \geq \mathrm{vc}(G) \geq 2$. If S is independent or $|S| = |X|$, then again $\mathrm{ivc}(G) = \mathrm{vc}(G)$.

Now we assume that S is not independent and $|X| > |S|$. This implies that there exist two adjacent vertices $x \in X \cap S$ and $y \in Y \cap S$, and another vertex $y' \in Y \setminus S$. By the structure of G, the vertex y' is adjacent to all but at most one vertex of X. Moreover, since $y' \notin S$, the neighbourhood of y' is contained in S. Therefore $|X| > |S| \geq |\{y\} \cup N(y')| \geq 1 + (|X| - 1) = |X|$, a contradiction. □

Theorem 1.4 (restated). *Let H be a graph. The class of H-free bipartite graphs is* ivc*-identical if and only if H is an induced subgraph of $K_{1,3}^{+}$ or of $2P_1+P_3$.*

Proof. First suppose that H is an induced subgraph of $K_{1,3}^{+}$ or of $2P_1 + P_3$. We start with the case where $H = K_{1,3}^{+}$. Let G be a $K_{1,3}^{+}$-free bipartite graph. We may assume without loss of generality that G is connected. By Lemma 2.1, G is either a path, a cycle or an almost complete bipartite graph. For the first two cases it is readily seen that $\mathrm{ivc}(G) = \mathrm{vc}(G)$. For the third case we apply Lemma 2.2.

Now suppose $H = 2P_1 + P_3$. Let G be a $(2P_1 + P_3)$-free bipartite graph with bipartite classes A and B, and let S be a minimum vertex cover of G. Suppose S is not an independent set. Then S contains two adjacent vertices x and y, say $x \in A$ and $y \in B$. Let I_x and I_y be the set of neighbours of x and y, respectively, in $G - S$. As S has minimum size, I_x and I_y are both nonempty. Moreover, as G is bipartite, $I_x \cap I_y = \emptyset$. As G is $(2P_1 + P_3)$-free, we find that $|I_x| \leq 1$ or $|I_y| \leq 1$, say $|I_x| \leq 1$. Let $I_x = \{u\}$. If $|I_y| \geq 2$, we replace S by $S' = (S \setminus \{x\}) \cup \{u\}$ to obtain another minimum vertex cover of G. Moreover, u has no neighbours in S'. In order to see this, let z be a neighbour of u in S', and let v_1, v_2 be two vertices in I_y. As $V(G) - S$ is an independent set, u is not adjacent to v_1 and v_2. As v_1, v_2, x, z all belong to A, they are also pairwise non-adjacent. Hence, the set $\{v_1, v_2, x, u, z\}$ induces $2P_1 + P_3$ in G, a contradiction. We conclude that replacing x by u yields a minimum vertex cover S' such that $G[S']$ contains at least one fewer edge than $G[S]$.

Let now S^* be a minimum vertex cover such that $G[S^*]$ has as few edges as possible. If S^* is independent, then we have proven that $\mathrm{ivc}(G) = \mathrm{vc}(G)$. Suppose S^* is not an independent set. Then S^* contains two adjacent vertices x^* and y^*, say $x^* \in A$ and $y^* \in B$. By the choice of S^* and the above discussion, we conclude that each of x^* and y^* has exactly one (private) neighbour in $G - S^*$. Since G is $(2P_1 + P_3)$-free, this means that $G - S^*$ has at most three vertices. The latter implies that at least one of $|A \setminus S^*|$ and $|B \setminus S^*|$, say $|A \setminus S^*|$, has at most one vertex. But now, since $|B \cap S^*| \geq 1$, we have $\mathrm{ivc}(G) \geq \mathrm{vc}(G) = |S^*| = |A \cap S^*| + |B \cap S^*| \geq |A \cap S^*| + |A \setminus S^*| = |A| \geq \mathrm{ivc}(G)$, and hence $\mathrm{ivc}(G) = \mathrm{vc}(G)$.

Now suppose that H is not an induced subgraph of $K_{1,3}^{+}$ or of $2P_1 + P_3$. By Theorem 1.1 we need only consider the case where H is an induced subgraph of $K_{1,r} + rP_1$ or $K_{1,r}^{+}$ for some $r \geq 1$. Hence, H contains an induced subgraph from the set $\{K_{1,4}, K_{1,3}+P_1, 3P_1+P_2, 5P_1\}$. Let G be the double star with two leaves for each central vertex, that is, G is the tree on vertices x, y, u_1, u_2, v_1, v_2 and edges xy, u_1x, u_2x, v_1y, v_2y. We note that G is bipartite and $(K_{1,4}, K_{1,3}+P_1, 3P_1+P_2, 5P_1)$-free and thus H-free, while $\mathrm{vc}(G) = 2$ and $\mathrm{ivc}(G) = 3$. This completes the proof. □

To prove Theorem 1.5 we will need the following lemma from **[11]**.

Lemma 2.3 ([**11**]). *If G is a (P_1+P_2)-free near-bipartite graph, then* $\mathrm{ifvs}(G) = \mathrm{fvs}(G)$.

Theorem 1.5 (restated). *Let $H \neq K_{1,3}$ be a graph. The class of H-free near-bipartite graphs is* ifvs-*identical if and only if H is a subgraph of P_3.*

Proof. First suppose that H is a subgraph of P_3. If $H = P_1+P_2$, then $\mathrm{ifvs}(G) = \mathrm{fvs}(G)$ for every H-free near-bipartite graph G by Lemma 2.3. If $H = P_3$, then every H-free near-bipartite graph G is a disjoint union of complete graphs on at most three vertices, and hence $\mathrm{ifvs}(G) = \mathrm{fvs}(G)$ holds. Finally suppose that $H = 3P_1$. Let G be a $3P_1$-free near-bipartite graph. As G is $3P_1$-free, any minimum independent feedback vertex set of G has size at most 2. Hence, any minimum feedback vertex set of G also has size at most 2. Moreover, if it has size 1, then it is an independent feedback vertex set. We conclude that $\mathrm{ifvs}(G) = \mathrm{fvs}(G)$.

Now suppose that H is not a subgraph of P_3. Recall that we assume that $H \neq K_{1,3}$. By Theorem 1.2 we may then assume that $H = K_{1,r}$ for some $r \geq 4$ or $H = sP_1$ for some $s \geq 4$. Consider the graph G in Figure 1. It is straightforward to check that G is $4P_1$-free and near-bipartite; $\{u, v\}$ is a minimum feedback vertex set (indeed $G - \{u, v\}$ is P_5) while $\mathrm{ifvs}(G) = 3$ (for instance, $\{u, u_1, u_2\}$ is a minimum independent feedback vertex set of G). This completes the proof. □

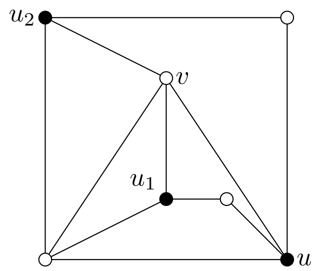

Figure 1. An example of a $4P_1$-free near-bipartite graph G with $\mathrm{ifvs}(G) = \mathrm{fvs}(G) + 1$.

3. Conclusions

We fully classified for which graphs H the class of H-free graphs is ivc-identical. We did the same for the notions of being ifvs-identical and being ioct-identical, only here a few cases remain open. We pose these as open problems.

Open Problem 3.1. Does there exist a $K_{1,3}$-free near-bipartite graph G with $\mathrm{ifvs}(G) > \mathrm{fvs}(G)$?

Open Problem 3.2. For $H \in \{K_{1,3}, K_{1,3}^{+}, 2P_1+P_3\}$, does there exist an H-free 3-colourable graph G with $\mathrm{ioct}(G) > \mathrm{oct}(G)$?

We note that the classes of $K_{1,3}$-free near-bipartite graphs and $K_{1,3}$-free 3-colourable graphs are ifvs-bounded and ioct-bounded by Theorems 1.2 and 1.3, respectively. However, it is also still open if the class of $K_{1,3}^{+}$-free 3-colourable graphs is ioct-bounded.

We also note that the classes of $K_{1,3}$-free near-bipartite graphs and $K_{1,3}$-free 3-colourable graphs are NP-complete to recognize. This follows from the results that the problems of deciding near-bipartiteness [**3**] and deciding 3-colourability [**14**] are NP-complete for line graphs, which form a subclass of $K_{1,3}$-free graphs.

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