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DEGREE CONDITIONS FORCING ORIENTED CYCLES

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ABSTRACT. The longstanding Caccetta-Häggkvist Conjecture is asking for the minimum outdegree (or semidegree) in an oriented graph that forces the appearance of a directed cycle of a bounded length. Motivated by this, Kelly, Kühn and Osthus made a conjecture on the minimal semidegree forcing the appearance of a directed cycle of a given length, and proved it for cycles of length not divisible by 3. Here we prove all the remaining cases of their conjecture with the optimal semidegree threshold.

1. INTRODUCTION

One of the most famous conjectures in graph theory is the following conjecture stated more than 40 years ago by Caccetta and Häggkvist [**4**].

Conjecture 1 (Caccetta-Häggkvist)**.** Every n-vertex oriented graph G with minimum outdegree $\delta^+(G) \geq \frac{n}{\ell}$ contains an oriented cycle of length at most ℓ.

Here, by an oriented graph we understand a directed graph without loops and multiple edges, i.e., an orientation of a simple graph. The conjecture was proved for many large values of ℓ [**4, 8, 10, 19**], with additive error term in the bound for the cycle length [**5, 15, 20**], with multiplicative error term in the outdegree assumption [**3, 4, 7, 9, 14, 18**], or with additional assumption on forbidden subgraphs [**6, 17**]. For more results and problems related to the Caccetta-Häggkvist Conjecture see a summary [**21**]. The weaker conjecture with assumption on the minimal semidegree (minimum of outdegrees and indegrees over all vertices) is also open and generalizes a conjecture of Behzad, Chartrand and Wall [**2**] from 1970.

Motivated by the above conjecture, Kelly, Kühn and Osthus [**12**] made the following conjecture.

Conjecture 2 (Kelly-Kühn-Osthus)**.** For any $\ell \geq 4$ every big enough n-vertex oriented graph G with $\delta^{\pm}(G) \geq \frac{n}{k} + \frac{1}{k}$ contains an oriented cycle of length exactly ℓ, where k is the smallest integer greater than 2 that does not divide ℓ.

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Kelly, Kühn and Osthus [**12**] proved it for $k = 3$, which means for ℓ not divisible by 3, and also showed the asymptotic version of the conjecture for $k = 4$ with $\ell \geq 42$ and for $k = 5$ with $\ell \geq 2550$. Additionally, they proved that a bound of $n/3 + 1$ suffices to force a cycle of length ℓ for any $\ell \geq 4$.

Later, Kühn, Osthus and Piguet [**13**] proved an asymptotic version of the conjecture for ℓ big enough, i.e., that for any $\ell \geq 10^7 k^6$ and $\varepsilon \geq 0$, every big enough oriented graph G with $\delta^{\pm}(G) > \frac{n}{k}(1+\varepsilon)$ contains a cycle of length ℓ.

2. Main results

We present constructions showing that in general the conjectured threshold is not correct, and we prove the optimal threshold for each $\ell \geq 4$.

Theorem 3. *For any $\ell \geq 4$ every big enough n-vertex oriented graph G with semidegree $\delta^{\pm}(G) \geq \frac{n}{k} + \frac{k-1}{2k}$ contains a directed cycle of length ℓ, where k is the smallest integer greater than 2 that does not divide ℓ.*

Moreover, if $\ell \not\equiv 3 \pmod{12}$, then this is the best possible threshold. If $\ell \equiv 3 \pmod{12}$, then $\delta^{\pm}(G) \geq \frac{n}{4} + \frac{1}{4}$ is already forcing an ℓ-cycle and this is the best possible threshold.

The proof for the case $k = 4$ is different than the proof for larger values of k. In particular, for $k \geq 5$ one can prove the following stability version, which is not true for $k = 4$.

Theorem 4. *For $\ell \geq 4$ let k being the smallest integer greater than 2 that does not divide ℓ. If $k \geq 5$, then any oriented graph H with $\delta^{\pm}(H) \geq \frac{n}{k}\left(1 - \frac{1}{30k}\right)$ that does not contain a closed walk of length ℓ, is a subgraph of a blow-up of k-cycle.*

3. Constructions

We show that the semidegree threshold in Theorem 3 is optimal separately for odd and even values of k.

For odd k, we need to present an n-vertex graph G without ℓ-cycle having semidegree $\delta^{\pm}(G) = \frac{n}{k} + \frac{k-3}{2k}$. To construct such a graph, we start with a balanced blow-up of a k-cycle on $n + (k-3)/2$ vertices. This way we have good semidegree and no ℓ-cycle. Now, we need to remove $(k-3)/2$ vertices, without changing the semidegree and avoiding creating an ℓ-cycle. In order to do this we use $(k-3)/2$ times one of the maneuvers described on Figure 1. For $\ell < k/2$ we use $(k-1)/2 - \ell$ times the first maneuver and $\ell - 1$ times the second maneuver. While for $\ell > k/2$ we use $k - 1 - \ell$ times the first maneuver and $\ell - (k+1)/2$ times the second maneuver. In both cases, we cannot create cycles of lengths congruent to $\ell \pmod{k}$.

When k is an even number then we need to present an n-vertex graph G without ℓ-cycle having semidegree $\delta^{\pm}(G) = \frac{n}{k} + \frac{k-2}{2k}$. Similarly as before, to construct such a graph, we start with a balanced blow-up of a k-cycle on $n + (k-2)/2$ vertices and we need to remove $(k-2)/2$ vertices without changing the semidegree and avoiding creating an ℓ-cycle.

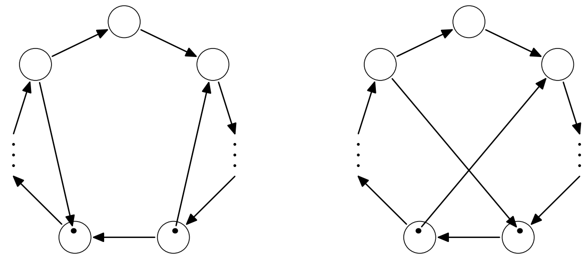

Figure 1. Maneuvers used for the constructions for odd k. By adding edges to arbitrarily chosen two vertices in the bottommost blobs, one can remove a vertex in the topmost blob.

If $\ell \equiv k/2 \pmod{k}$, which covers all the considered cases except $k = 4$ and $\ell \not\equiv 2 \pmod{4}$, then we can use $(k-2)/2$ times the maneuver described on Figure 2.

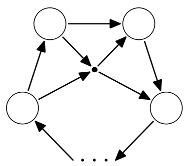

Figure 2. Maneuver used for the constructions for even k. By adding the middle vertex and incident edges, one can remove a vertex in each topmost blob.

In the case $k = 4$ and $\ell \equiv 3 \pmod{4}$, the optimal construction for Theorem 3 is just a blow-up of a 4-cycle. In the case $k = 4$ and $\ell \equiv 1 \pmod{4}$, one can consider a blow-up of a 4-cycle with all diagonal edges to a single vertex in one blob, and all diagonal edges from a single vertex in a non-adjacent blob.

4. Overview of the proofs

In order to prove Theorem 4, we firstly obtain that such a graph has bounded directed diameter by providing bounds on the sizes of neighborhoods. This can be achieved by proving the Caccetta-Häggkvist Conjecture with multiplicative error term in the outdegree assumption, using similar ideas as in [**20**]. Then we show using Frobenius coin problem that such a graph cannot contain short cycles with one edge reversed. This is enough to define the wanted blobs and prove the theorem by analysis of the graph structure.

To prove Theorem 3 for $k \geq 5$, we use the regularity lemma for oriented graphs by Alon and Shapira [**1**] and Theorem 4, to obtain a structure of the graph G with some small fraction of additional edges and vertices. Then, by application of some results on additive combinatorics, we prove that the assumed semidegree threshold gives enough many additional edges and vertices to construct the wanted cycle of length ℓ.

The case of $k = 4$ requires a different approach. In particular, notice that for $\ell = 6$ one blob of a 4-cycle blow-up can contain arbitrary one-way oriented bipartite graph. Also, for any odd ℓ, one can reverse edges of any 4-cycle contained in a blow-up of a 4-cycle keeping the semidegree assumption and avoiding cycles of odd lengths.

Our proof of the main theorem for $k = 4$ needs directed diameter bound, that cannot be obtained in the same way as for $k \geq 5$. To achieve this, we use the method of flag algebras created by Razborov [**16**]. The proofs, especially in the cases of $\ell = 6$ and $\ell = 9$, need also more detailed analysis, because the small length of the wanted cycle can cause complications in combining some partial structures to obtain the cycle of length ℓ.

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