Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 827–834

DUSHNIK-MILLER DIMENSION OF STAIR CONTACT COMPLEXES

L. ISENMANN and D. GONÇALVES

Abstract. The theorem of Schnyder asserts that a graph is planar if and only if the Dushnik-Miller dimension of its poset of incidence is at most 3. Trotter aksed how this can be generalized to higher dimensions. Towards this goal, Dushnik-Miller dimension has been studied in terms of TD-Delaunay complexes, in terms of orthogonal surfaces, and in terms of polytopes. Here we consider the relation between the Dushnik-Miller dimension and contact systems of stairs in $\mathbb{R}^d$. We propose two different definitions of stairs in $\mathbb{R}^d$ which are connected to the Dushnik-Miller dimension as follows. The first definition allows us to characterize supremum sections, which are simplicial complexes related to the Dushnik-Miller dimension, in two different ways. The second definition provides for any Dushnik-Miller dimension at most $d+1$ complex a representation as a contact system of stairs in $\mathbb{R}^d$.

1. Intoduction

The Dushnik-Miller dimension (also known as the order dimension) of a poset P has been introduced by Dushnik and Miller [**3**]. It is the minimum number of linear extensions of P such that P is the intersection of these extensions. See [**16**] for a comprehensive study of this topic. Schnyder [**15**] studied the Dushnik-Miller dimension of the incidence posets of graphs which captures the fact that a vertex is incident to some edges. Some classes of graphs can be characterized by their *Dushnik-Miller* dimension which we define as the Dushnik-Miller dimension of their incidence poset. For example, path forests are the graphs of Dushnik-Miller dimension at most 2. Schnyder [**15**] obtained a celebrated combinatorial characterization of planar graphs: they are those of Dushnik-Miller dimension at most 3. The question of characterizing classes of graphs of larger dimension is open [**16**]. Nevertheless there are some partial results. Ossona de Mendez [**13**] (see also [**1**]) showed that every simplicial complex of Dushnik-Miller dimension d has a straight line embedding in $\mathbb{R}^{d-1}$ which generalizes the result of Schnyder in a way. The reciprocal is false by considering K_n which has a straight line embedding in $\mathbb{R}^3$ while its Dushnik-Miller dimension is $\Theta(\log \log n)$ [**11**]. The class of graphs of Dushnik-Miller dimension at most 4 is rather rich. Extremal questions in this class of graphs have been studied: Felsner and Trotter [**8**] showed

Received June 7, 2019.
2010 *Mathematics Subject Classification.* Primary 06A06; Secondary 05C62.
This research is partially supported by the ANR GATO, under contract ANR-16-CE40-0009.

that these graphs can have a quadratic number of edges. Furthermore, in order to solve a question about conflict free coloring [**5**], Chen *et al.* [**2**] showed that many graphs of Dushnik-Miller dimension 4 only have independent sets of size at most $o(n)$. This result also implies that there is no constant k such that every graph of Dushnik-Miller dimension at most 4 is k-colorable. Therefore, graphs of Dushnik-Miller dimension at most 4 seem difficult to characterize. Nevertheless, it was conjectured in [**12**] (see also [**4**]) that the class of Dushnik-Miller dimension d complexes is the class of TD-Delaunay complexes. But this conjecture is false [**9**]. Following the work of Scarf [**14**], Felsner and Kappes [**7**], considered the notion of Dushnik-Miller dimension through the lens of orthogonal surfaces.

In [**10**], the authors studied contact graphs of shapes similar to thick ⌞'s in the plane. They proved that every triangle free or 4-connected planar graph can be obtained as contact graphs of thick ⌞'s. Thick ⌞'s can be seen as particular stairs where there is only one bend. Allowing more general stairs one obtains all planar graphs as contact graphs. We consider two ways to define stairs in $\mathbb{R}^2$. The first one consists in taking the union of rectangles which have the same bottom left corner. The second one consists in taking a positive quadrant and removing other positive quadrants. This two definitions are equivalent and can be generalized to higher dimensions as we will see: the first one leads to the notion of *stairs* and the second one to *truncated stairs*.

Motivated by Trotter's question, this observation leads us to introduce in this work the notion of stairs and of truncated stairs in $\mathbb{R}^d$ which can be seen as the generalization of thick ⌞'s in the plane. We study the link between contact complexes of stairs in $\mathbb{R}^d$ and the Dushnik-Miller dimension. In the second section, we prove that supremum sections are in bijection with contact complexes of truncated stairs defined from an ordered point set. In the third section, we prove that supremum sections are in bijection with contact complexes of truncated stairs which have the property to be in some sense stable under the operation consisting in removing vertices. In the fourth section, we prove that any simplicial complexes of Dushnik-Miller dimension $d+1$ can be represented as the contact complex of stairs in $\mathbb{R}^d$.

2. Truncated stair contact complexes

In the following, V is a finite set. We denote by $\mathsf{Subsets}(V)$ the set of subsets of V. An *(abstract) simplicial complex* Δ is a subset of $\mathsf{Subsets}(V)$ closed by inclusion (i.e., $\forall X \in \Delta, \forall Y \subseteq X, Y \in \Delta$). We call *faces* the elements of Δ and *facets* the maximal faces of Δ according to the inclusion order.

Definition 2.1 (Ossona de Mendez [**13**])**.** Given a linear order $\leq$ on a set V, an element $x \in V$, and a set $F \subseteq V$, we say that x *dominates* F in $\leq$, and we denote it $F \leq x$, if $f \leq x$ for every $f \in F$. A *d-representation* R on a set V is a set of d linear orders $\leq_1, \dots, \leq_d$ on V. Given a d-representation R, an element $x \in V$, and a set $F \subseteq V$, we say that x *dominates* F in R if x dominates F in some order $\leq_i \in R$. We define $\Sigma(R)$ as the set of subsets F of V such that every $v \in V$ dominates F in R. The set $\Sigma(R)$ is called the *supremum section* of R.

The Dushnik-Miller dimension of a simplicial complex is defined as the Dushnik-Miller dimension of its inclusion poset. Ossona de Mendez [**13**] proved that a simplicial complex Δ is of Dushnik-Miller dimension at most d if and only if there exists a d-representation R on its vertex set V such that $\Delta \subseteq \Sigma(R)$. It is easy to show that if R is a d-representation on a set V, then $\Sigma(R)$ is a simplicial complex. An example is the following 3-representation on $\{a, b, c, d, e\}$:

$$a <_1 b <_1 e <_1 d <_1 c$$
$$c <_2 b <_2 a <_2 d <_2 e$$
$$e <_3 d <_3 c <_3 b <_3 a$$

The corresponding complex $\Sigma(R)$ is characterized by its facets $\{a, b\}, \{b, c, d\}$, and $\{b, d, e\}$. For example $\{a, b, c\}$ is not in $\Sigma(R)$ as b does not dominate $\{a, b, c\}$ in any order.

For any finite set F of points in $\mathbb{R}^d$, we define the point $p^F \in \mathbb{R}^d$ as follows: $p^F_i = \max_{x\in F} x_i$ for every $i \in [\![1, d]\!]$. Let $p \in \mathbb{R}^d$, we define $I_p = \{z \in \mathbb{R}^d : p_i \leq z_i \ \forall i \in [\![1, d]\!]\}$ and $\mathring{I}_p = \{z \in \mathbb{R}^d : p_i < z_i \ \forall i \in [\![1, d]\!]\}$. In other words, I_p denotes the (*closed*) *positive orthant* of $\mathbb{R}^d$ whose corner is p and $\mathring{I}_p$ denotes its open version. Given a subset A of $\mathbb{R}^d$, A^c denotes the complementary of A in $\mathbb{R}^d$. Given a set of points $\mathcal{P}$ of $\mathbb{R}^d$, we say that $\mathcal{P}$ is in *general position* if no two points of $\mathcal{P}$ share the same coordinate for any direction. In the rest of this article, points will always be in general position.

Definition 2.2. A *truncated stair system* $\mathcal{S}$ is given by a set $\mathcal{P}$ of points of $\mathbb{R}^d$ and a function $\mathcal{F} : \mathcal{P} \to \mathsf{Subsets}(\mathcal{P})$. Let $x \in \mathcal{P}$, we define the (closed) *truncated stair* $S(x, \mathcal{P}, \mathcal{F})$ and the *open truncated stair* $\mathring{S}(x, \mathcal{P}, \mathcal{F})$ of x according to the truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ by

$$S(x, \mathcal{P}, \mathcal{F}) = I_x \bigcap_{z\in\mathcal{F}(x)} (\mathring{I}_z)^c \qquad\qquad \mathring{S}(x, \mathcal{P}, \mathcal{F}) = \mathring{I}_x \bigcap_{z\in\mathcal{F}(x)} (I_z)^c$$

When there is no confusion possible, we would write $S(x)$ instead of $S(x, \mathcal{P}, \mathcal{F})$. Remark furthermore that $S(x, \mathcal{P}, \mathcal{F})$ is closed and that $\mathring{S}(x, \mathcal{P}, \mathcal{F})$ is open for every $x \in \mathcal{P}$.

We say that a truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is a *truncated stair packing* if for every pair (x, y) of distinct elements of $\mathcal{P}$, $\mathring{S}(x, \mathcal{P}, \mathcal{F}) \bigcap \mathring{S}(y, \mathcal{P}, \mathcal{F}) = \emptyset$. Given a truncated stair packing $\mathcal{S}$, we define the contact complex $\Delta(\mathcal{S})$ of $\mathcal{S}$ as the abstract simplicial complex on the vertex set $\mathcal{P}$ where $F \subseteq \mathcal{P}$ is a face of $\Delta(\mathcal{S})$ if $\bigcap_{x\in F} S(x, \mathcal{P}, \mathcal{F}) \neq \emptyset$. We say that a truncated stair packing $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is a *truncated stair tiling* if for every $x \in \mathcal{P}$, $I_x \subseteq \bigcup_{y\in\mathcal{P}} S(y, \mathcal{P}, \mathcal{F})$.

A truncated stair packing can be seen as an arrangement of non-overlapping truncated stairs. An example is given in Figure 1.

Lemma 2.3. *If* $F \in \Delta(\mathcal{P}, \mathcal{F})$*, then* $p^F \in \bigcap_{x\in F} S(x, \mathcal{P}, \mathcal{F})$.

Due to lack of space, complete proofs are not provided in this extended abstract.

According to the previous lemma, we have a different characterization of faces of $\Delta(\mathcal{P}, \mathcal{F})$. Indeed $F \in \Delta(\mathcal{P}, \mathcal{F})$ if and only if $p^F \in \bigcap_{x\in F} S(x, \mathcal{P}, \mathcal{F})$.

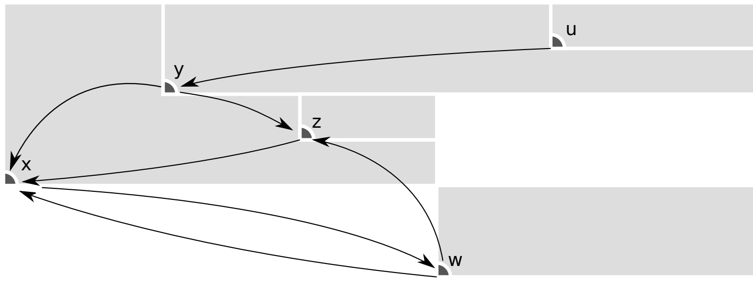

Figure 1. An example of a truncated stair packing. An arc from y to x means that $y \in \mathcal{F}(x)$.

Definition 2.4. Let $\mathcal{P}$ be a set of points of $\mathbb{R}^d$ and $\leq$ a linear order on $\mathcal{P}$. We define the function $\mathcal{F}_{\leq} : \mathcal{P} \to \mathsf{Subsets}(\mathcal{P})$ by $\mathcal{F}_{\leq}(x) = \{z \in \mathcal{P} : z < x\}$. We define the *ordered truncated stair system* $(\mathcal{P}, \leq)$ as the truncated stair system $(\mathcal{P}, \mathcal{F}_{\leq})$. We say that a truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is an *ordered truncated stair system* if there exists an order $\leq$ on $\mathcal{P}$ such that $\mathcal{S} = (\mathcal{P}, \leq)$.

Remark that ordered truncated stair systems are particular cases of truncated stair systems. These kind of truncated stair systems have good property and we will see that they are connected to supremum sections.

Lemma 2.5. *Any ordered truncated stair system is a truncated stair tiling.*

3. Dushnik-Miller dimension of ordered truncated stair systems

Lemma 3.1. *Let $\mathcal{P}$ be a set of points of $\mathbb{R}^d$ and let $R = (\leq_1, \dots, \leq_{d+1})$ be a $(d+1)$-representation on $\mathcal{P}$. If for every p and $q \in \mathcal{P}$ and every $i \in [\![1, d]\!]$, $p_i \leq q_i \iff p \leq_i q$, then $\Sigma(R) = \Delta(\mathcal{P}, \leq_{d+1})$.*

Proof. Let $F \in \Sigma(R)$. We define the point p as p^F. Let us check that $p^F \in \bigcap_{x \in F} S(x)$. Let $x \in F$. As $x_i \leq \max_{u \in F} u_i = p_i^F$ for every i, then $p^F \in I_x$. Let $z \in \mathcal{P}$ such that $z <_{d+1} x$. Suppose by contradiction that $p^F \notin (\mathring{I}_z)^c$. Then $p^F \in \mathring{I}_z$ and $z_i < p_i^F = \max_{u \in F} u_i$ for all $i \in [\![1, d]\!]$. Thus for every i, there exists $u \in F$ such that $z <_i u$. As $z <_{d+1} x$, we conclude that z does not dominate F in R which contradicts the fact that $F \in \Sigma(R)$. Therefore $p^F \in (\mathring{I}_z)^c$ and $p^F \in S(x)$. We conclude that $\bigcap_{x \in F} S(x)$ is not empty and thus $F \in \Delta(\mathcal{P}, \leq_{d+1})$.

Consider a set $F \subseteq \mathcal{P}$ such that $\bigcap_{x \in F} S(x)$ is not empty and $p \in \bigcap_{x \in F} S(x)$. Towards a contradiction, consider a point z that does no dominate F in R. There exists $x \in F$ such that $z <_{d+1} x$. As $p \in S(x)$ and $z <_{d+1} x$, then $p \in (\mathring{I}_z)^c$. Then there exists $i \in [\![1, d]\!]$ such that $z_i \geq p_i$. For every $y \in F$, $p \in I_y$ and thus $p_i \geq y_i$. We conclude that $z_i \geq \max_{u \in F} u_i$. Thus $z \geq_i u$ for every $u \in F$ and z dominates F in the order $\leq_i$, a contradiction. We conclude that $F \in \Sigma(R)$. □

With the help of Lemma 3.1, we prove the following theorem.

Theorem 3.2. *Let Δ be a complex on a vertex set V. Then Δ is the contact complex of an ordered truncated stair system of $\mathbb{R}^d$ if only if there exists a $(d+1)$-representation R on V such that $\Delta = \Sigma(R)$.*

4. Dushnik-Miller dimension of truncated stair tilings

The work done for ordered truncated stair tilings allows us to study truncated stair tilings. Nevertheless, the characterization proven here needs another property, called *removal property.*

Definition 4.1. Let $(\mathcal{P}, \mathcal{F})$ be a truncated stair system. The truncated stair system $(\mathcal{P} \smallsetminus \{x\}, \mathcal{F}')$ obtained after the removal of an element x of $\mathcal{P}$ is defined as follows:

$$\mathcal{F}'(y) = \begin{cases} \mathcal{F}(y) \bigcup \mathcal{F}(y) \smallsetminus \{x\} & \text{if } x \in \mathcal{F}(y) \\ \mathcal{F}(y) & \text{otherwise} \end{cases}$$

We say that the truncated stair tiling $(\mathcal{P}, \mathcal{F})$ has the *removal property* if any truncated stair system obtained after a sequence of removal is still a truncated stair tiling.

Lemma 4.2. *Let $\mathcal{P}$ be a set of points of $\mathbb{R}^d$ and $\leq$ be a linear order on $\mathcal{P}$. Then the truncated stair system $(\mathcal{P}, \leq) = (\mathcal{P}, \mathcal{F}_{\leq})$ has the removal property.*

Lemma 4.3. *Let $(\mathcal{P}, \mathcal{F})$ be a truncated stair tiling which has the removal property. Then there exists a partial order $\leq$ on $\mathcal{P}$ such that $\Delta(\mathcal{P}, \mathcal{F}) = \Delta(\mathcal{P}, \leq)$.*

Sketch of proof. We define a binary relation $\leq$ on $\mathcal{P}$ defined by $x \leq y$ if and only if $x \in \mathcal{F}(y)$. Let us show that this relation is acyclic. Suppose that there exists a cycle C in this relation. We define $(C, \mathcal{F}')$ the truncated stair system obtained after the removal of elements not in C. By the removal property of $(\mathcal{P}, \mathcal{F})$, the truncated stair tiling $(C, \mathcal{F}')$ is a truncated stair tiling. Consider the point $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$ defined by: $p_i = 1 + \max_{u \in C} u_i$ for every $i \in [\![1, d]\!]$. As $(C, \mathcal{F}')$ is a truncated stair tiling and as $p \in I_x$ for every $x \in C$, then there exists $x \in C$ such that $p \in S(x, \mathcal{P}, \mathcal{F}')$. As C is a cycle, there exists $y \in C$ such that $y \in \mathcal{F}'(x)$. Thus $p \in (\mathring{I}_y)^c$: there exists $i \in [\![1, d]\!]$ such that $p_i \leq y_i$. This contradicts the definition of p. We conclude that the relation $\leq$ is acyclic.

By considering any transitive closure $\leq'$ of $\leq$, we show that $S(y, \mathcal{P}, \mathcal{F}) = S(y, \mathcal{P}, \leq')$ for every $y \in \mathcal{P}$. □

The removal property is important in this proof as there exists truncated stair tilings which have not the removal property. It is for instance the case of the following truncated stair system: $x = (3, 0, 1)$, $y = (1, 3, 0)$, $z = (0, 1, 3)$ and $w = (2, 2, 2)$ where the function $\mathcal{F}$ is defined by $\mathcal{F}(x) = \{y, w\}$, $\mathcal{F}(y) = \{z, w\}$, $\mathcal{F}(z) = \{x, w\}$ and $\mathcal{F}(w) = \emptyset$.

Thanks to Lemma 4.2, Lemma 2.5 and Theorem 3.2, we can prove the following theorem.

Theorem 4.4. *Let Δ be a complex on a vertex set V. Then Δ is the contact complex of a truncated stair tiling of $\mathbb{R}^d$ which has the removal property if only if there exists a $(d+1)$-representation R on V such that $\Delta = \Sigma(R)$.*

5. Dushnik-Miller dimension and stair systems

We introduce here a variant of the definition of stair systems. This one is more geometrical as it does not need an order or a function like in the previous definitions.

Let $y \in \mathbb{R}^d$, we define the negative orthant J_y and the open negative orthant $\mathring{J}_y$ which are subsets of $\mathbb{R}^d$ as follows $J_y = \{u \in \mathbb{R}^d : u_i \leq y_i \ \forall i \in [\![1, d]\!]\}$ and $\mathring{J}_y = \{u \in \mathbb{R}^d : u_i < y_i \ \forall i \in [\![1, d]\!]\}$.

Definition 5.1. A *stair sytem* $\mathcal{S}$ is given by a set $\mathcal{P}$ of points of $\mathbb{R}^d$ and a function $C \colon \mathcal{P} \to \mathsf{Subsets}(\mathbb{R}^d)$ such that $C(x)$ is finite for every $x \in \mathcal{P}$. Let $x \in \mathcal{P}$, we define the (closed) *stair* $R(x, \mathcal{P}, C)$ and the *open stair* $\mathring{R}(x, \mathcal{P}, C)$ of x according to the stair system $\mathcal{S} = (\mathcal{P}, C)$ by

$$R(x, \mathcal{P}, C) = \bigcup_{y \in C(x)} I_x \bigcap J_y \qquad \mathring{R}(x, \mathcal{P}, C) = \bigcup_{y \in C(x)} \mathring{I}_x \bigcap \mathring{J}_y$$

We say that $(\mathcal{P}, C)$ is a *stair packing* if the sets $\mathring{R}(x, \mathcal{P}, C)$ are disjoint.

When there will be no confusion possible, we would write $R(x)$ instead of $R(x, \mathcal{P}, C)$. An example of a stair packing is given in Figure 2.

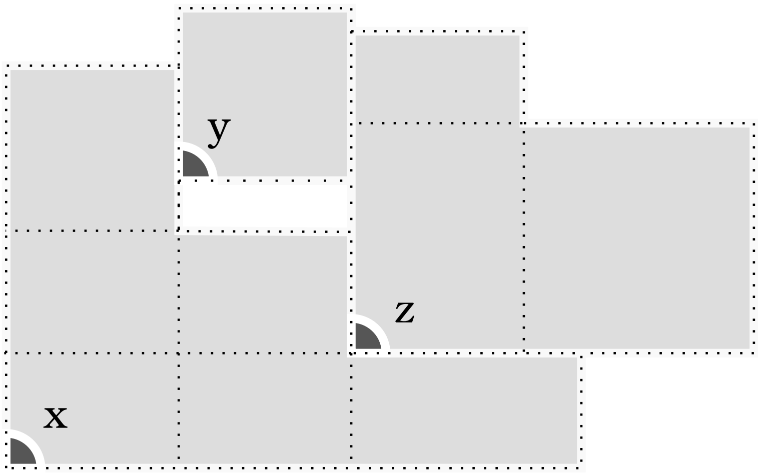

Figure 2. An example of a stair packing.

Definition 5.2. Let $(\mathcal{P}, C)$ be a stair packing. We define the contact complex of $\mathcal{P}$, denoted $\Delta(\mathcal{P}, C)$ as follows: a subset F of $\mathcal{P}$ is a face of $\Delta(\mathcal{P}, C)$ if and only if $\bigcap_{x \in F} R(x)$ is not empty.

Remark that $\Delta(\mathcal{P}, C)$ is a simplicial complex. Furthermore remark that this definition generalizes the previous definition of a truncated stair system.

Theorem 5.3. *Let Δ be a simplicial complex of Dushnik-Miller dimension at most $d+1$. Then Δ is the contact complex of a stair system in $\mathbb{R}^d$.*

Sketch of proof. There exists a $d+1$-representation $R = (\leq_1, \dots, \leq_{d+1})$ on V, the vertex set of Δ, such that $\Delta \subseteq \Sigma(R)$. According to Lemma 3.1, there exists an embedding $\mathcal{P}$ of V such that $\mathcal{P}$ is in general position and such that $\Sigma(R) = \Delta(\mathcal{P}, \leq_{d+1})$.

Let us define a stair on $\mathcal{P}$ as follows. For every vertex $x \in \mathcal{P}$, we define $R(x) = \bigcup_{F\in\Delta:\, x\in F} I_x \bigcap J_{p^F}$. We show by contradiction that $(\mathcal{P}, C)$ is a stair packing. We conclude by proving by double inclusion that $\Delta = \Delta(\mathcal{P}, C)$. □

6. Conclusion

According to Theorem 5.3, we conclude that the class of Dushnik-Miller dimension at most $d+1$ complexes is included in the class of contact complexes of stair packing in $\mathbb{R}^d$. The reciprocal of this theorem is a question that arises naturally. Finally, the case where the point set $\mathcal{P}$ is not in general position, in all the three results, seems interesting but more involved.

References

1. Bayer D., Peeva I. and Sturmfels B., *Monomial resolutions*, Math. Res. Lett. **5** (1998), 31–46.
2. Chen X., Pach J., Szegedy M. and Tardos G., *Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles*, Random Structures Algorithms **34** (2009), 11–23.
3. Dushnik B. and Miller E. W., *Partially ordered sets*, Amer. J. Math. **63** (1941), 600–610.
4. Evans W., Felsner S., Kobourov G. and Ueckerdt T., *Graphs admitting d-realizers: spanning-tree-decompositions and box-representations*, in: Proc. of EuroCG 14, 2014.
5. Even G., Lotker Z., Ron D. and Smorodinsky S., *Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks*, SIAM J. Comput. **33** (2003), 94–136.
6. Felsner S., *Empty rectangles and graph dimension*, `arXiv:math/0601.767`
7. Felsner S. and Kappes S., *Orthogonal surfaces and their CP-orders*, Order **25** (2008), 19–47.
8. Felsner S. and Trotter W. T., *Posets and planar graphs*, J. Graph Theory **49** (2005), 262–272.
9. Gonçalves D. and Isenmann L., *Dushnik-Miller dimension of TD-Delaunay complexes*, in: Proc. of EuroCG 17, `arXiv:math/1803.09576`.
10. Gonçalves D., Isenmann L. and Pennarun C., *Planar graphs as L-intersection or L-contact graphs*, in: Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, 2018, 172–184.
11. Hoşten S. and Morris W. D., *The order dimension of the complete graph*, Discrete math. **201** (1999), 133–139.
12. Mary A., *Dimension de Poset et Subdivisions Simpliciales*, Master Thesis, Univ. of Montpellier, 2010.
13. Ossona de Mendez P., *Geometric realization of simplicial Complexes*, in: Proc. of Int. Symp. on Graph Drawing 1999 LNCS 1731, 1999, 323–332.
14. Scarf H., *The Computation of Economic Equilibria*, Cowles Foundation Monograph, vol. 24, Yale University Press, 1973.

15. Schnyder W., *Planar graphs and poset dimension*, Order **5** (1989), 323–343.
16. Trotter W. T., *Combinatorics and Partially Ordered Sets: Dimension Theory*, Johns Hopkins Series in the Mathematical Sciences, The Johns Hopkins University Press, 1992.

L. Isenmann, LIRMM, Université de Montpellier & CNRS, France,
e-mail: lucas.isenmann@lirmm.fr

D. Gonçalves, LIRMM, Université de Montpellier & CNRS, France,
e-mail: daniel.goncalves@lirmm.fr