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# MAJORITY COLORING OF INFINITE DIGRAPHS

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Abstract. Let $D$ be a finite or infinite digraph. A *majority coloring* of $D$ is a vertex coloring such that at least half of the out-neighbors of every vertex $v$ have different color than $v$. Let $\mu(D)$ denote the least number of colors needed for a majority coloring of $D$. It is known that $\mu(D) \leq 4$ for any finite digraph $D$, and $\mu(D) \leq 2$ if $D$ is acyclic. We prove that $\mu(D) \leq 5$ for any countably infinite digraph $D$, and $\mu(D) \leq 3$ if $D$ does not contain finite directed cycles. We also state a twin supposition to the famous Unfriendly Partition Conjecture.

## 1. Introduction

A *majority coloring* of a graph $G$ is a vertex coloring such that the neighbors of any vertex $v$ colored differently than $v$ constitute at least half of all the neighbors of $v$. In other words, at least half of the edges incident to $v$ are properly colored. The least number of colors needed for a majority coloring of $G$ is denoted as $\mu(G)$.

A folklore result in graph theory asserts that every finite graph $G$ satisfies $\mu(G) \leq 2$. The proof is very simple: just take a 2-coloring that minimizes the number of monochromatic edges (see [**7**]). For infinite graphs the problem is much harder. It was proved by Shelah and Milner [**9**] that every infinite graph $G$ satisfies $\mu(G) \leq 3$, and that there are uncountable graphs for which equality holds. However, the *Unfriendly Partition Conjecture* states that every countable graph satisfies $\mu(G) \leq 2$ (see [**1**]). It has been confirmed in several special cases, in particular when $G$ does not contain an infinite path [**4**], or a subdivision of an infinite clique [**3**],[**4**].

Majority coloring of directed graphs was introduced recently in [**6**]. Definition is the same, except that the majority condition concerns *out-neighbors* of vertices. Formally, a coloring of digraph $D$ is a *majority coloring* if the out-neighbors of any vertex $v$ colored differently than $v$ constitute at least half of all the out-neighbors of $v$. It was proved in [**6**] that $\mu(D) \leq 4$ for every finite digraph $D$. The proof is also quite easy. First, notice that every *acyclic* digraph (with no directed cycles) is majority 2-colorable (apply greedy coloring to linear ordering of the vertices of $D$ in which all out-neighbors of a given vertex are to the left). Next, split the edges of $D$ into two acyclic digraphs, and then take the product coloring. It is conjectured in [**6**] that every finite digraph $D$ satisfies $\mu(D) \leq 3$. This is best possible since

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a majority coloring of an odd directed cycle must be a proper coloring of the underlying undirected graph.

In this paper we study majority coloring of countably infinite digraphs. Our main results asserts that $\mu(D) \leq 5$ for any countable digraph $D$, and $\mu(D) \leq 3$ if $D$ is *acyclic* (does not contain finite directed cycles). In analogy to the Unfriendly Partition Conjecture for undirected graphs we state the following conjecture for digraphs.

**Conjecture 1.** Every countable digraph $D$ satisfies $\mu(D) \leq 3$. If $D$ is acyclic, then $\mu(D) \leq 2$.

Other open problems are presented in the final section of the paper.

## 2. Results

We will need the following lemma.

**Lemma 2.1.** *Let $D$ be a locally finite acyclic digraph. Suppose that each vertex $v$ is assigned with a pair of two real numbers $(r_v(1), r_v(2))$ satisfying condition:*

$$r_v(1) + r_v(2) \geq d^+(v). \tag{1}$$

*Then there is a coloring $c : V(D) \to \{1, 2\}$ such that for every vertex $v$, the color $c(v)$ appears on at most $r_v(c(v))$ out-neighbors of $v$. In particular, every locally finite acyclic digraph $D$ satisfies $\mu(D) \leq 2$.*

*Proof.* First we prove the lemma for finite acyclic digraphs by induction on the number of vertices of $D$. It is obvious that the assertion holds for a single vertex. Let $D$ be a digraph with at least two vertices, and let $u$ be a vertex with no in-neighbors. Let $D' = D - u$ be a digraph obtained from $D$ by deleting vertex $u$. Notice that $D'$ satisfies condition (1) with the same numbers $r_v(i)$, with $v \neq u$, $i = 1, 2$. So, we may apply induction to $D'$ to get a coloring satisfying assertion of the lemma. Let $n_u(1)$ and $n_u(2)$ denote the number of out-neighbors of $u$ in color 1 and 2 in this coloring, respectively. By condition (1), either $n_u(1) \leq r_u(1)$, or $n_u(2) \leq r_u(2)$. Hence, by choosing appropriate color for vertex $u$, we may extend the coloring to whole digraph $D$.

The assertion for infinite digraphs follows easily by compactness. A majority 2-coloring of $D$ is obtained when $r_v(1) = r_v(2)$ for every vertex $v$. $\square$

We now deal with an opposite case, when all vertices in a digraph $D$ have infinite out-degrees. We prove a slightly more general lemma.

**Lemma 2.2.** *Let $V$ be a countable set, and let $A_1, A_2, \ldots$ be a collection of infinite subsets of $V$. There is a 2-coloring of $V$ such that each color appears infinitely many times in each set $A_i$. In particular, every countable digraph in which all vertices have infinite out-degree is majority 2-colorable.*

*Proof.* We will make use of the "back-and-forth" argument, similar to the one asserting that the set of rationals is countable. Let $V = \{v_1, v_2, \ldots\}$ be any numeration of the elements of set $V$. This enumeration induces a linear order on every set $A_i$ in a natural way. Consider the sequences $S = 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$ consisting of finite blocks of initial positive integers. Let $s_i$ denote the $i$-th term of sequence $S$. We will color the elements of $V$ in consecutive steps, accordingly to the sequences $S$, as follows. In the first step we enter the first set $A_1$ and color its first element by color 1. In each subsequent step $s_i$ we enter the set $A_{s_i}$ and color the first uncolored vertex with a color different than the color applied last time we entered $A_{s_i}$. Notice that after each step the number of colored vertices in each set $A_i$ is finite. Also, the procedure guarantees that each set $A_i$ is entered infinitely many times. Hence, in each set $A_i$ the number of color changes is infinite. This proves the first part of the lemma. Second part follows easily by taking as $A_i$'s the family of closed out-neighborhoods of vertices in a given digraph. □

Now we are ready to prove one of the aforementioned results. We will need some notation. Let $D$ be a digraph on countable set of vertices $V$. For a vertex $v \in V$, let $N^+(v)$ denote the set of out-neighbors of $v$. Denote by $d^+(v)$ the cardinality of the set $N^+(v)$. For a subset $X$ of $V$ and any vertex $v \in V$, denote by $d^+_X(v)$ the cardinality of the set $N^+(v) \cap X$.

**Theorem 2.3.** *Every countable acyclic digraph $D$ satisfies $\mu(D) \leq 3$.*

*Proof.* Let $D$ be a digraph on countable set of vertices $V$. Let $F$ be the subset of $V$ consisting of all vertices with $d^+(v)$ finite, and let $I = V \setminus F$. Next, split the set $I$ into two subset sets $I = A \cup B$ as follows: $A = \{v \in I : d^+_F(v) = \infty\}$ and $B = I \setminus A$. So, each vertex $v \in B$ satisfies $d^+_I(v) = \infty$.

Now we define a coloring of $V$ by colors $\{1, 2, 3\}$ in the following way. First, color every vertex $v \in A$ by color 3. The rest of vertices will be colored by colors $1, 2$, so majority condition for vertices $v \in A$ will be satisfied by $d^+_F(v) = \infty$. Next consider a family of sets $B_v = \{v\} \cup (N^+(v) \cap B)$ indexed by those vertices $v \in B$ for which $B_v$ is infinite. We color the union of sets $B_v$ by $1, 2$ accordingly to Lemma 2.2. Hence majority condition is satisfied for these vertices $v$. If some vertices of $B$ are left uncolored, color them arbitrarily by $1, 2$. For such vertices majority condition is also satisfied, since each of them has infinitely many out-neighbors in the set $A$ (which is colored by 3). We are left with completing the coloring on the set $F$. We will use Lemma 2.1. Let $n_v(i)$ denote the number of out-neighbors of $v \in F$ in color $i$, where $i = 1, 2, 3$. Put

$$r_v(i) = \frac{1}{2}d^+(v) - \frac{1}{2}n_v(3) - n_v(i), \tag{2}$$

for $i = 1, 2$. Clearly, we have $r_v(1) + r_v(2) = d^+_F(v)$, so we may produce a coloring of a subdigraph induced by $F$ satisfying the assertion of Lemma 2.1. This coloring guarantees that majority condition is satisfied for each vertex $v \in F$. The proof is complete. □

Now we prove a similar result for general countable digraphs. The proof goes along similar lines. We will use the following lemma, which is a special case of the result in [**2**]. We include a proof for completeness.

**Lemma 2.4.** *Let $D$ be a locally finite digraph. Suppose that each vertex $v$ is assigned with a quadruple of real numbers $(r_v(1), r_v(2), r_v(4), r_v(4))$ satisfying condition:*

$$r_v(1) + r_v(2) + r_v(3) + r_v(4) \geq 2d^+(v). \tag{3}$$

*Then there is a coloring $c\colon V(D) \to \{1, 2, 3, 4\}$ such that for every vertex $v$, the color $c(v)$ appears on at most $r_v(c(v))$ out-neighbors of $v$. In particular, every locally finite digraph $D$ satisfies $\mu(D) \leq 4$.*

*Proof.* Let us call the number $r_v(i)$ the *rank* of color $i$ at vertex $v$. In finite case the proof goes by induction on the number of vertices in $D$. It is not hard to check that the theorem is true for one-vertex digraph. Indeed, by condition (3), at least one color rank is non-negative, and we may use it to color the only vertex in the digraph. So, let $n \geq 2$, and assume that the assertion of the theorem is true for all digraphs with at most $n-1$ vertices. Let $D$ be a digraph on $n$ vertices satisfying the assumptions of the theorem. Let $u$ be any vertex of $D$. Consider a new digraph $D'$ obtained by deleting vertex $u$ with color ranks $r'_v(i)$, $v \neq u$, modified as follows. Let $a$ and $b$ be two colors with highest ranks $r_u(a)$, $r_u(b)$ at vertex $u$. For each in-coming neighbor $v$ of $u$, decrease the ranks $r_v(a)$ and $r_v(b)$ by one, so $r'_v(i) = r_v(i) - 1$ for $i \in \{a, b\}$. All the remaining color ranks are left unchanged: $r'_v(i) = r_v(i)$ if $i \notin \{a, b\}$, or if $v$ is not an in-coming neighbor of $u$.

We claim that digraph $D'$ with modified color ranks $r'_v(i)$ still satisfies condition (3). Indeed, for each in-coming neighbor $v$ of $u$, both sides of (3) decreased by exactly two (since the out-degree $d^+(u)$ decreased by exactly one). In all other cases nothing changed. So, by the inductive assumption there is a coloring of $D'$ satisfying the assertion of the theorem with ranks $r'_v(i)$.

We now extend this coloring to the deleted vertex $u$ in the following way. First notice that

$$r_u(a) + r_u(b) \geq d^+(u). \tag{4}$$

Indeed, by the maximality of ranks of colors $a$ and $b$ at vertex $u$, the inequality $r_u(a) + r_u(b) < d^+(u)$ would imply $\sum_{i \in \{1,2,3,4\}} r_u(i) < 2d^+(u)$, contrary to the assumption. Let $n_u(a)$ and $n_u(b)$ denote the number of out-neighbors of $u$ colored with colors $a$ and $b$, respectively. Obviously, $n_u(a) + n_u(b) \leq d^+(u)$. Hence, by (4), at least one of the following inequalities must be satisfied:

$$r_u(a) \geq n_u(a) \quad \text{or} \quad r_u(b) \geq n_u(b). \tag{5}$$

We choose a color whose rank satisfies one of these inequalities, and assign that color to $u$.

We claim that the extended coloring satisfies the assertion of the theorem. First, let $v$ be arbitrary in-coming neighbor of $u$. Let $x$ denote the color assigned to $v$ in coloring of $D'$. If $x$ is one of the colors $a$ or $b$, then the number of out-neighbors of $v$ in $D'$ colored with $x$ is at most $r'_v(x) = r_v(x) - 1$, by inductive assumption. Thus, their number in $D$ after coloring the vertex $u$ is still bounded by $r_v(x)$. If $x$ is neither equal to $a$, nor to $b$, then the constraint is fulfilled even more. If $v$ is an arbitrary out-neighbor of $u$, or any other vertex of $D'$, then the corresponding constraint holds by induction, since out-neighborhoods and color ranks for such vertices remained unchanged in $D'$. Finally, for the vertex $u$ we have chosen color $a$ or $b$ so that the corresponding inequality of (5) is satisfied. This completes the proof in finite case.

If $D$ is infinite, the assertion follows by compactness. A majority 4-coloring of $D$ is obtained when $r_v(1) = r_v(2) = r_v(3) = r_v(4)$ for every vertex $v$. □

Using Lemma 2.4 we may now prove an upper bound for $\mu(D)$ for general countable digraphs.

**Theorem 2.5.** *Every countable digraph $D$ satisfies $\mu(D) \leq 5$.*

*Proof.* The reasoning goes similarly as in the proof of Theorem 2.3. With the same notation, let $D$ be a countable digraph on vertex set $V$, and let $F$ be the subset of $V$ consisting of all vertices with finite out-degree $d^+(v)$. Let $I = V \setminus F$ be split similarly into two subsets $I = A \cup B$, where $A = \{v \in I : d_F^+(v) = \infty\}$ and $B = I \setminus A$. So, each vertex $v \in B$ satisfies $d_I^+(v) = \infty$.

Define a coloring of $V$ by colors $\{1, 2, 3, 4, 5\}$ in the following way. First, color every vertex $v \in A$ by color 5. The rest of vertices will be colored by colors $1, 2, 3, 4$, so majority condition for vertices $v \in A$ will be satisfied by $d_F^+(v) = \infty$. Next color the set $B$ as in the proof of Theorem 2.3 by colors $1, 2$ using Lemma 2.2. Clearly, majority condition is satisfied for all vertices $v \in B$. To complete the coloring on the set $F$ we use Lemma 2.4. Let $n_v(i)$ denote the number of out-neighbors of $v \in F$ in color $i$, where $i = 1, 2, 3, 4, 5$. Put

$$r_v(i) = \frac{1}{2}d^+(v) - \frac{1}{4}n_v(5) - n_v(i), \tag{6}$$

for $i = 1, 2, 3, 4$. Clearly, we have $r_v(1) + r_v(2) + r_v(3) + r_v(4) = d_F^+(v)$, so we may produce a coloring of a subdigraph induced by $F$ satisfying the assertion of Lemma 2.4. This coloring guarantees that majority condition is satisfied for each vertex $v \in F$. The proof is complete. □

## 3. Discussion

We conclude the paper with some remarks and open problems. It is natural to expect that, similarly as it is for undirected graphs, Theorem 2.5 remains true for arbitrary infinite digraphs.

**Conjecture 2.** Every infinite digraph satisfies $\mu(D) \leq 5$.

Suppose that each vertex $v$ of a digraph $D$ is assigned a set (*list*) of colors $L(v)$. A vertex coloring $c$ of digraph $D$ is a coloring *from lists* $L(v)$ if $c(v) \in L(v)$ for every vertex $v$. Let $\mu_\ell(D)$ denote the least cardinal $k$ such that digraph $D$ has a majority coloring from arbitrary lists of size $k$. Clearly, $\mu(D) \leq \mu_\ell(D)$. In [**2**] we proved that $\mu_\ell(D) \leq 4$ for every finite digraph $D$. For infinite digraphs we do not know if there is any finite upper bound.

**Conjecture 3.** There is an integer $K$ such that $\mu_\ell(D) \leq K$ for every countable digraph $D$.

A similar conjecture can be stated for undirected graphs.

**Conjecture 4.** There is an integer $N$ such that $\mu_\ell(G) \le N$ for every countable graph $G$.

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