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POWERS OF HAMILTONIAN CYCLES IN μ-INSEPARABLE GRAPHS

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ABSTRACT. We consider sufficient conditions for the existence of k-th powers of Hamiltonian cycles in n-vertex graphs G with minimum degree μn for arbitrarily small $\mu > 0$. About 20 years ago Komlós, Sárközy, and Szemerédi resolved the conjectures of Pósa and Seymour and obtained optimal minimum degree conditions for this problem by showing that $\mu = \frac{k}{k+1}$ suffices for large n. Consequently, for smaller values of μ the given graph G must satisfy additional assumptions. We show that inducing subgraphs of density $d > 0$ on linear subsets of vertices and being inseparable, in the sense that every cut has density at least $\mu > 0$, are sufficient assumptions for this problem. This generalises a recent result of Staden and Treglown.

1. INTRODUCTION AND NEW RESULT

We study sufficient conditions for the existence of powers of Hamiltonian cycles in large finite graphs. For $k \in \mathbb{N}$ the *k-th power* of a given graph H is the graph H^k on the same vertex set with xy being an edge in H^k if x and y are two vertices of H that are connected in H by a path of at most k edges. In a k-th power of a cycle every $k+1$ consecutive vertices (consecutive in the underlying cyclic ordering of the cycle) span a clique K_{k+1}. In particular, if a graph $G = (V, E)$ contains the k-th power of a Hamiltonian cycle, it also contains $\lfloor \frac{|V|}{k+1} \rfloor$ pairwise vertex disjoint copies of K_{k+1} and G contains a K_{k+1}-factor if $|V|$ is divisible by $k+1$.

Establishing sufficient conditions for the existence of Hamiltonian cycles in graphs has a long history and Dirac's well known theorem [**3**] yields a best possible minimum degree condition for this problem. The minimum degree of a graph turned out to be an interesting parameter for enforcing a given spanning subgraph and establishing optimal minimum degree conditions for those problems turned out to be a fruitful research direction in extremal graph theory (see, e.g., [**1**] and the references therein). Already about 50 years ago the minimum degree problem

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for K_{k+1}-factors was resolved by Corrádi and Hajnal [**2**] for $k = 2$ and by Hajnal and Szemerédi [**5**] for every $k \geq 3$. Pósa (see [**4**]) and Seymour [**11**] asked for a common generalisation of those results on factors and Dirac's theorem and conjectured that the best possible minimum degree conditions for K_{k+1}-factors and k-th powers of Hamiltonian cycles are the same (given that the number of vertices is divisible by $k + 1$). The general conjecture was affirmatively resolved for sufficiently large graphs by Komlós, Sárközy, and Szemerédi [**6**] by establishing the following result.

Theorem 1.1 (Komlós, Sárközy & Szemerédi 1998)**.** *For every integer $k \geq 1$ there exists n_0 such that if G is a graph on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq \frac{k}{k+1}n$, then G contains the k-th power of a Hamiltonian cycle.* □

Note that for $k = 1$ we recover Dirac's theorem (up to the value of n_0) and complete and nearly balanced $(k + 1)$-partite graphs show that the minimum degree condition in Theorem 1.1 is best possible for every k. Those lower bound constructions are ruled out by restricting the independence number of the large graph G. Here we consider the following robust restriction that imposes a uniformly positive edge density for induced subgraphs on linear sized subsets of vertices.

Definition 1.2. We say that a graph $G = (V, E)$ is *(ϱ, d)-dense* for $\varrho > 0$ and $d \in [0, 1]$ if $e(X) \geq d\binom{|X|}{2} - \varrho|V|^2$ for every subset $X \subseteq V$, where $e(X)$ denotes the number of edges of G that are contained in X.

In fact, it was shown by Staden and Treglown [**12**] that such a hereditary density assumption for any $d > 0$ and sufficiently small $\varrho > 0$ allows us to reduce the minimum degree condition for k-th powers of Hamiltonian cycles to

$$\delta(G) \geq \Big(\frac{1}{2} + \mu\Big)|V|, \tag{1}$$

for any $\mu > 0$ and sufficiently large vertex sets $V = V(G)$ (see also [**9**] for K_{k+1}-factors). In particular, the minimum degree assumption becomes independent of k. Moreover, the graph G consisting of two disjoint cliques on close to $n/2$ vertices (one of them with the number of vertices not divisible by $k + 1$) shows that this degree condition is essentially optimal for guaranteeing clique factors or powers of Hamiltonian cycles. However, a bipartite version of Definition 1.2, which requires

$$e(X, Y) = \big|\{(x, y) \in X \times Y : xy \in E(G)\}\big| \geq d\,|X||Y| - \varrho|V|^2\,, \tag{2}$$

for all subsets X, $Y \subseteq V$, rules out this example. It was observed by Glock and Joos (see [**12**, Concluding Remarks]) that imposing property (2) on G allows a further relaxation of the minimum degree condition for G from (1) to $\mu|V|$ for arbitrary $\mu > 0$. We show that requiring property (2) for all subsets X and Y is not needed. It already suffices to assume it only for vertex bipartitions of G.

Definition 1.3. We say that a graph $G = (V, E)$ is *μ-inseparable* for some $\mu > 0$ if $e(X, V \smallsetminus X) \geq \mu\,|X||V \smallsetminus X|$ for every subset $X \subseteq V$.

Invoking this assumption to subsets X consisting of one vertex only, yields a linear minimum degree condition for μ-inseparable graphs G.

Our new result asserts that graphs satisfying the properties of Definitions 1.2 and 1.3 contain k-th powers of Hamiltonian cycles for every fixed integer $k \geq 1$.

Theorem 1.4 (Main result)**.** *For every integer $k \geq 1$ and all constants $d \in [0,1]$ and $\mu > 0$, there exist $\varrho > 0$ and n_0 such that the following holds.*

If $G = (V,E)$ is a (ϱ, d)-dense and μ-inseparable graph with $|V| = n \geq n_0$ vertices, then G contains the k-th power of a Hamiltonian cycle.

The proof of Theorem 1.4 utilises the absorption method of Rödl, Ruciński, and Szemerédi [**10**], which, roughly speaking, consists of the following three parts:

1. finding an almost perfect cover of only "few" k-th powers of paths,
2. ensuring the abundant existence of so-called *absorbers*, and
3. connecting those absorbers and paths to an almost spanning k-th power of a cycle.

For the first and second part of the absorption method our proof relies only on the (ϱ, d)-denseness assumption, while the third part is based on both assumptions and uses ideas from [**8**] and we omit the details here.

It is easy to see that every graph $G = (V,E)$ with minimum degree at least $\delta(G) \geq (1/2 + \mu)|V|$ is μ-inseparable and, consequently, Theorem 1.4 strengthens the result of Staden and Treglown for powers of Hamiltonian cycles [**12**].

Moreover, the n-vertex graph G obtained from two cliques of size $(1/2 + \mu)n$ which intersect in $2\mu n$ vertices is $(\varrho, 1/2)$-dense (for any fixed $\varrho > 0$), while it fails to satisfy property (2) for arbitrary subsets X and Y. This shows that our result is not covered by the observation of Glock and Joos [**12**, Concluding Remarks].

On the other hand, we remark that Staden and Treglown and also Glock and Joos not only considered the embedding problem for powers of Hamiltonian cycles, but more generally for bounded degree graphs (with and without small bandwidth, see [**12**] for details). Theorem 1.4 can also be used to obtain such a strengthening for a variant of the bandwidth theorem under the same assumptions (see the full version of the manuscript for details).

2. Open problems

An interesting direction of research might be to find a common generalisation of the approximate version of Theorem 1.1 and of Theorem 1.4. More precisely, this would require to weaken the assumptions of Theorem 1.4 in such a way that, on one hand, every n-vertex graph with minimum degree $(\frac{k}{k+1} + o(1))n$ would satisfy them and, on the other hand, they still ensure the existence of a k-th power of a Hamiltonian cycle.

For $k = 1$, i.e., for the existence of Hamiltonian cycles, such a common generalisation can be obtained by relaxing the (ϱ, d)-denseness condition only for sets $X \subseteq V$ of size at least $|X| > |V|/2$, while for smaller sets we require that either they induce $d|X|^2/2 - \varrho n^2$ edges or there are at least $(1-\varrho)|X|$ vertices outside X with at least $d|X| - \varrho n$ neighbours in X.

In fact, for an absorption-type proof for the existence of Hamiltonian cycles we remark that μ-inseparability alone suffices to build the connection of the pieces from the almost perfect path cover and from the absorbers in this case. Moreover, a combination of some ideas from [**7**] and [**8**] shows that μ-inseparability also yields absorbers for such an approach as long as there are no independent sets of size $n/2$. The relaxation of the (ϱ, d)-denseness assumption outlined above still guarantees almost perfect matchings in a "robust" way, which can be utilised to guarantee an almost perfect path cover (for details we refer to the full version of the manuscript).

For higher powers ($k \geq 2$) of Hamiltonian cycles we remark that μ-inseparability does not suffice to ensure the existence of absorbers. For example, random bipartite graphs of edge density $2\mu + o(1)$ are μ-inseparable, while absorbers for k-th powers of Hamiltonian cycles must contain cliques of order $k+1$. However, the main challenge seems to be to find a useful condition that guarantees almost perfect K_{k+1}-factors in a "robust" way.

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