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BIJECTIONS FOR GENERALIZED TAMARI INTERVALS VIA ORIENTATIONS

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Abstract. We introduce two bijections for generalized Tamari intervals, which were recently introduced by Préville-Ratelle and Viennot, and proved to be in bijection with rooted non-separable maps by Fang and Préville-Ratelle. Our first construction proceeds via separating decompositions on quadrangulations and can be seen as an extension of the Bernardi-Bonichon bijection between Tamari intervals and minimal Schnyder woods. Our second construction directly exploits the Bernardi-Bonichon bijection and the point of view of generalized Tamari intervals as a special case of classical Tamari intervals (synchronized Tamari intervals); it yields a trivariate generating function expression that interpolates between generalized Tamari intervals and classical Tamari intervals.

1. Introduction

The ν-Tamari lattice $\mathrm{Tam}(\nu)$ (for ν an arbitrary directed walk with steps in $\{N, E\}$) has been recently introduced by Préville-Ratelle and Viennot [**16**], and further studies in [**7, 8**], with connections to geometric combinatorics. It is a lattice on the set of directed walks weakly above ν and with same endpoints as ν, and it generalizes the Tamari lattice [**18**] (in size n, case where ν is NE replicated n times) and the m-Tamari lattices [**1**] (in size n, case where ν is NE^m replicated n times).

The enumeration of intervals (i.e., pairs formed by two elements x, x' with $x \leq x'$) in Tamari lattices has attracted a lot of attention [**9, 5, 4**], due in particular to their (conjectural) connections to dimensions of diagonal coinvariant spaces [**1**], and to their bijective connections to planar maps [**2**], as well as intriguing symmetry properties [**15**]. Regarding ν-Tamari lattices, if we let $\mathcal{I}_\nu$ be the set of intervals in $\mathrm{Tam}(\nu)$, then it has recently been shown by Fang and Préville-Ratelle [**11**] that $\mathcal{G}_n := \cup_{\nu \in \{N,E\}^n} \mathcal{I}_\nu$ (generalized Tamari intervals of size n) is in bijection with the set $\mathcal{N}_n$ of rooted non-separable maps[1] with $n+2$ edges,

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[1] A *map* is a connected multigraph embedded on the sphere up to deformation, a *rooted map* is a map with a marked corner, and a map is called *non-separable* (or 2-connected) if it is either the loop-map, or has no loop and at least 2 vertices need to be deleted to disconnect it.

and more precisely that $\mathcal{G}_{i,j} := \sum_{\nu \in \mathfrak{S}(N^i E^j)} \mathcal{I}_\nu$ is in bijection with the set $\mathcal{N}_{i,j}$ of rooted non-separable maps with $i+2$ vertices and $j+2$ faces (it is known [**6**] that $|\mathcal{N}_n| = \frac{2(3n+3)!}{(n+2)!(2n+3)!}$ and $|\mathcal{N}_{i,j}| = \frac{(2i+j+1)!(2j+i+1)!}{(i+1)!(j+1)!(2i+1)!(2j+1)!}$). They have a first recursive bijection based on parallel decompositions with a catalytic variable, and then make the bijection more explicit via certain auxiliary labeled trees.

In this article, we give two new bijections between $\mathcal{G}_{i,j}$ and $\mathcal{N}_{i,j}$. Each one relies on seeing $\mathcal{G}_{i,j}$ as included in a certain superfamily, and specializing a bijection involving oriented maps. In our first bijection (Section 3) we see $\mathcal{G}_{i,j}$ as a subfamily of non-intersecting triples of lattice paths (a so-called Baxter family) and specialize a bijection (closely related to the one in [**14**]) with so-called separating decompositions on rooted simple quadrangulations. In our second bijection (Section 4) we see $\mathcal{G}_{i,j}$ as a subfamily of classical Tamari intervals of size $i+j+1$ (synchronized intervals), to which we apply the Bernardi-Bonichon bijection [**2**] (with minimal Schnyder woods, on rooted triangulations) combined with a bijection [**3**] to certain tree-structures on which we can characterize the property of being synchronized.

Several parameters can be tracked by the first construction, which gives a model of maps for intervals in the m-Tamari lattices, and reveals certain symmetry properties on $\mathcal{G}_{i,j}$. The second construction yields a trivariate generating function expression that interpolates between the bivariate generating function of generalized Tamari intervals and the univariate generating function of classical Tamari intervals (see Corollary 1 and paragraph after).

2. The ν-Tamari lattice, and generalized Tamari intervals

We recall [**16**] the definitions of ν-Tamari lattices and intervals, and how they relate to the classical Tamari lattice. We consider walks in $\mathbb{N}^2$ starting at the origin and having steps North or East (these can be identified with words on the alphabet $\{N, E\}$). For two such walks γ, γ', we say that γ' is above γ if γ and γ' have the same endpoint, and no East step of γ is strictly above the East step of γ' in the same vertical column. A Dyck walk of length $2n$ is thus a walk γ that is above $(NE)^n$. More generally, for ν a walk ending at (i,j), we let $\mathcal{W}_\nu$ be the set of walks above ν. For $\gamma \in \mathcal{W}_\nu$ and for $p = (x,y)$ a point on γ, we let $x' \geq x$ be the abscissa of the North step of ν from ordinate y to $y+1$ (with the convention that $x' = i$ if $y = j$), and we let $\ell(p) := x' - x$. If p is preceded by E and followed by N we let p' be the next point after p along γ such that $\ell(p') = \ell(p)$, and we let $\mathrm{push}_p(\gamma)$ be the walk γ' obtained from γ by moving the E preceding p to be just after p' (see Figure 1 for an example); we say that γ' covers γ. The Tamari lattice for ν is defined as $\mathrm{Tam}(\nu) = (\mathcal{W}_\nu, \leq)$ where $\leq$ is the transitive closure of the covering relation. The classical Tamari lattice Tam_n corresponds to the special case $\mathrm{Tam}_n = \mathrm{Tam}((NE)^n)$, and more generally for $m \geq 1$ the m-Tamari lattice $\mathrm{Tam}_n^{(m)}$ corresponds to the special case $\mathrm{Tam}_n^{(m)} = \mathrm{Tam}((NE^m)^n)$.

Interestingly, for ν of length n, $\mathrm{Tam}(\nu)$ can also be obtained as a sublattice of Tam_{n+1}. For $\gamma = E^{\alpha_0} N E^{\alpha_1} \dots N E^{\alpha_n}$ a Dyck walk of length $2n$, the *canopy-word* of γ is the word $\mathrm{can}(\gamma) = (w_0, \dots, w_n) \in \{0,1\}^{n+1}$ such that for $r \in [0..n]$,

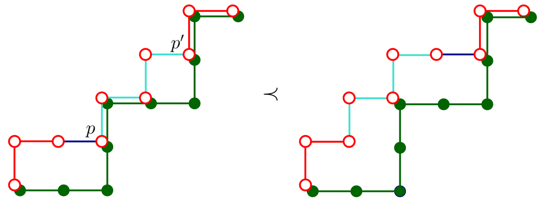

Figure 1. A covering relation in $\mathrm{Tam}(\nu)$ for $\nu = EENNEENNE$.

$w_r = \mathbf{1}_{\alpha_r=0}$ (note that we always have $w_0 = 1$ and $w_n = 0$). Then $\mathrm{Tam}(\nu)$ identifies to the sublattice of Tam_{n+1} induced by the Dyck walks whose canopy-word is equal to $0\nu1$ (this crucially relies on the fact that if $\gamma \leq \gamma'$ in Tam_n then $\mathrm{can}(\gamma) \leq \mathrm{can}(\gamma')$).

Note that $\mathcal{G}_{i,j}$ is the set of triples (ν, γ, γ') such that ν ends at (i, j), and $\gamma \leq \gamma'$ in $\mathrm{Tam}(\nu)$; By a classical correspondence from maps to quadrangulations, $\mathcal{N}_{i,j}$ is in bijection with the set $\mathcal{Q}_{i,j}$ of rooted bicolored quadrangulations with $i+2$ black vertices and $j+2$ white vertices (a quadrangulation is a simple map with all faces of degree 4, bicolored means that vertices are black or white so that all edges connect vertices of different color, and the vertex at the root-corner is black).

We now make two important remarks based on properties shown in [**16**] (each remark is associated with a bijection described later, respectively in Section 3 and Section 4, the second remark is also used in the bijection in [**11**]).

(i) Since $\gamma \leq \gamma'$ in $\mathrm{Tam}(\nu)$ implies that γ' is above γ, $\mathcal{G}_{i,j}$ is a subfamily of the family $\mathcal{R}_{i,j}$ of triples of walks (ν, γ, γ') such that ν ends at (i, j), and γ' is above γ itself above ν.

(ii) On the other hand, let $\mathcal{I}_n$ be the set of intervals in Tam_n (classical Tamari intervals, on Dyck words). An interval $(\gamma, \gamma') \in \mathcal{I}_n$ is called *synchronized* if $\mathrm{can}(\gamma) = \mathrm{can}(\gamma')$. Let $\mathcal{S}_n \subset \mathcal{I}_n$ be the set of synchronized Tamari intervals of size n. Then the above sublattice characterization of $\mathrm{Tam}(\nu)$ implies that $\mathcal{G}_n$ is in bijection with $\mathcal{S}_{n+1}$. More generally, if we let $\mathcal{S}_{i,j}$ be the set of synchronized intervals such that the common canopy-word has $i+1$ zeros and $j+1$ ones, then $\mathcal{G}_{i,j}$ is in bijection with $\mathcal{S}_{i,j}$.

3. Bijection using separating decompositions

Several bijections are known between $\mathcal{R}_{i,j}$ and other combinatorial families (a survey is given in [**13**]). Our aim here is to pick one such bijection and show that it specializes nicely to the subfamily $\mathcal{G}_{i,j} \subset \mathcal{R}_{i,j}$. We pick the bijection from [**14**] for separating decompositions, but have to slightly modify it so that it specializes well.

For $Q \in \mathcal{Q}_{i,j}$, we let s, s', t, t' be the outer vertices in clockwise order around the outer face, with s the one at the root. A *separating decomposition* of Q is given by an orientation and coloration (blue or red) of each edge of Q such that all edges incident to s (resp. t) are ingoing blue (resp. ingoing red), and

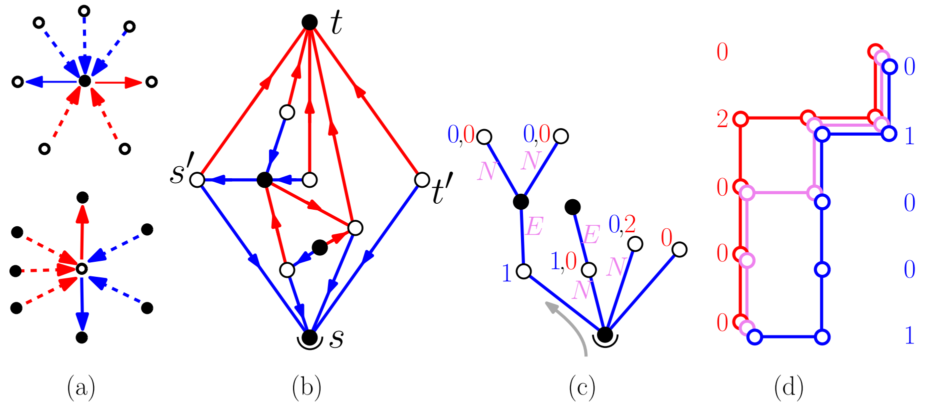

Figure 2. (a) Local rule of separating decompositions for vertices $\notin \{s,t\}$. (b) A separating decomposition. (c) The blue tree with the indication of red and blue indegrees at white vertices. (d) The corresponding (by Φ') triple of walks.

every vertex $v \notin \{s,t\}$ has outdegree 2 and satisfies the local conditions shown in Figure 2(a). It can be shown [**10**] that the blue edges form a spanning tree of $Q\backslash t$ and the red edges form a spanning tree of $Q\backslash s$. We let $\mathrm{Sep}_{i,j}$ be the set of pairs $S=(Q,X)$ where $Q\in\mathcal{Q}_{i,j}$, and X is a separating decomposition of Q. A separating decomposition is called *minimal* if it has no clockwise cycle. A general property of outdegree-constrained orientations of planar maps [**12**] ensures that each rooted quadrangulation has a unique minimal separating decomposition, so that $\mathcal{Q}_{i,j}$ identifies to the subfamily of minimal separating decompositions from $\mathrm{Sep}_{i,j}$. We now recall the bijection Φ between $\mathrm{Sep}_{i,j}$ and $\mathcal{R}_{i,j}$ described in [**14**]. For $S\in\mathrm{Sep}_{i,j}$ we let T_{blue} be the blue tree, and let $v_0,\ldots,v_{j+1}$ be the white vertices ordered according to the first visit in a clockwise walk around T_{blue} starting at the root, and we let α_k be the number of ingoing red edges at v_k, for $k\in[1..j+1]$.

Then $\Phi(S)$ is the triple of walks $(\gamma_{\mathrm{low}},\gamma_{\mathrm{mid}},\gamma_{\mathrm{up}})$ (written here as binary words) as follows: the walk γ_{low} is obtained from a clockwise walk around T_{blue}, where we write an N each time – except for the last two occurences – we follow an edge $\circ - \bullet$ getting closer to the root and write an E each time we follow an edge $\circ - \bullet$ away from the root; the walk γ_{mid} is obtained from a clockwise walk around T_{blue}, where we write an N each time – except for the first and last occurence – we follow an edge $\bullet - \circ$ away from the root and write an E each time we follow an edge $\bullet - \circ$ getting closer to the root; the walk γ_{up} is $E^{\alpha_1}NE^{\alpha_2}\ldots NE^{\alpha_{j+1}}$.

We slightly modify the mapping as follows (see Figure 2 for an example): $\Phi'(S)$ is the triple $(\gamma_{\mathrm{low}},\gamma_{\mathrm{mid}},\gamma_{\mathrm{up}})$ of walks where γ_{mid} and γ_{up} are obtained as above, and $\gamma_{\mathrm{low}}=E^{\beta_0}NE^{\beta_1}\ldots NE^{\beta_j}$, with β_r the number of ingoing blue edges at v_r for $r\in[0..j]$.

Theorem 1. *For $i,j\geq 0$, the mapping Φ' is a bijection between $\mathrm{Sep}_{i,j}$ and $\mathcal{R}_{i,j}$. In addition, for $S\in\mathrm{Sep}_{i,j}$, S is minimal iff $\Phi'(S)\in\mathcal{G}_{i,j}$, hence Φ' specializes into a bijection between $\mathcal{Q}_{i,j}$ and $\mathcal{G}_{i,j}$.*

This bijection easily yields a linear time random generator for $\mathcal{G}_{i,j}$ (using a bijection between $\mathcal{Q}_{i,j}$ and unrooted ternary trees [**17, 3**]), and it also has the advantage to preserve several parameters.

Further parameter-correspondence. Precisely, for $(\nu,\gamma,\gamma') \in \mathcal{R}_{i,j}$ and $r \in \{0..j\}$ we let a_r (resp. b_r) be the number of horizontal steps of ν (resp. γ') at height r. For $(p,q) \in \mathbb{N}^2$ we let $m(p,q) := \#\{r \in [1..j] \mid b_{r-1} = p \text{ and } a_r = q\}$. Then clearly in the bijection Φ', a_0 is mapped to the degree of s' minus 2, b_j is mapped to the degree of t' minus 2, and $m(p,q)$ corresponds to the number of inner white vertices that have p ingoing red edges and q ingoing blue edges.

A model of maps for intervals in $\mathrm{Tam}_n^{(m)}$**.** In particular, for $m \geq 1$, if we let $\mathcal{Q}_n^{(m)}$ be the subfamily of $\mathcal{Q}_{mn,n}$ where each inner white vertex has m ingoing blue edges in the minimal separating decomposition, and s' has no ingoing blue edge, then Φ' specializes into a bijection between $\mathcal{Q}_n^{(m)}$ and intervals of $\mathrm{Tam}_n^{(m)}$. It is known [**5**] (the case $m = 1$ was discovered in [**9**]) that the number $I_n^{(m)}$ of intervals in $\mathrm{Tam}_n^{(m)}$ is given by the beautiful formula

$$I_n^{(m)} = \frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}. \tag{1}$$

It is shown in [**14**] that $\mathcal{Q}_n^{(1)}$ is in bijection (via contraction of the blue edges directed toward a white vertex) with rooted triangulations (simple planar maps with all faces of degree 3) with $n+3$ vertices, endowed with their minimal Schnyder wood. Under this correspondence one can check that our bijection coincides with the one by Bernardi and Bonichon [**2**] (recalled and exploited in the next section) between $\mathcal{I}_n$ and rooted triangulations with $n+3$ vertices. It would be interesting to provide a bijective proof of (1), working for all $m \geq 1$, based on such an approach (edge-contractions or similar operations to obtain maps or hypermaps amenable to bijective enumeration).

An involution on $\mathrm{Sep}_{i,j}$ **and** $\mathcal{G}_{i,j}$**.** There is a natural involution τ on $\mathrm{Sep}_{i,j}$: move the root-corner to t and switch edge-colors. Via Φ', τ induces an involution on $\mathcal{R}_{i,j}$ whose effect is to swap a_0 and b_j and to swap $m(p,q)$ and $m(q,p)$. Note that τ preserves minimality, hence it induces an involution on $\mathcal{G}_{i,j}$, which reveals some symmetry properties. In particular, for $\lambda = 1^{\ell_1}2^{\ell_2}\ldots, \mu = 1^{m_1}2^{m_2}\ldots$ two integer partitions of i in at most $j+1$ parts, if we let $g_{i,j}[a,b,\lambda,\mu]$ be the number of elements in $\mathcal{G}_{i,j}$ such that $a_0 = a$, $b_j = b$, and for each $k \geq 1$ there are ℓ_k (resp. m_k) values $r \in [1..j]$ such that $a_r = k$ (resp. $b_{r-1} = k$), then by the involution we have $g_{i,j}[a,b,\lambda,\mu] = g_{i,j}[b,a,\mu,\lambda]$. In particular there are as many intervals in $\mathrm{Tam}_n^{(m)}$ as intervals $(\gamma_{\mathrm{low}},\gamma_{\mathrm{mid}},\gamma_{\mathrm{up}})$ in $\mathcal{G}_{mn,n}$ such that $\gamma_{\mathrm{up}} = (E^m N)^n$.

4. Bijection using Schnyder woods

For T a rooted triangulation, the outer vertices are called u_B, u_G, u_R in clockwise order, with u_B the one incident to the root-corner. A *Schnyder wood* of T is an orientation and coloration (in blue, green or red) of every inner edge of T so that all edges incident to u_B, u_G, u_R are ingoing of color blue (resp. green, red), and every inner vertex has outdegree 3 and satisfies the local condition shown in Figure 3. A Schnyder wood induces a coloring of the corners: a corner at an inner vertex v receives the color of the 'opposite' outgoing edge at v, and a corner at an outer vertex v receives the color of v. It can be checked that around each inner face there is one corner in each color and these occur as blue, green, red in clockwise order. It is known that the local conditions of Schnyder woods imply that the graph in every color is a tree spanning all the internal vertices (plus the outer vertex of the same color, the root-vertex of the tree). A Schnyder wood is called *minimal* if it has no clockwise cycle; any rooted triangulation has a unique minimal Schnyder wood [**12**].

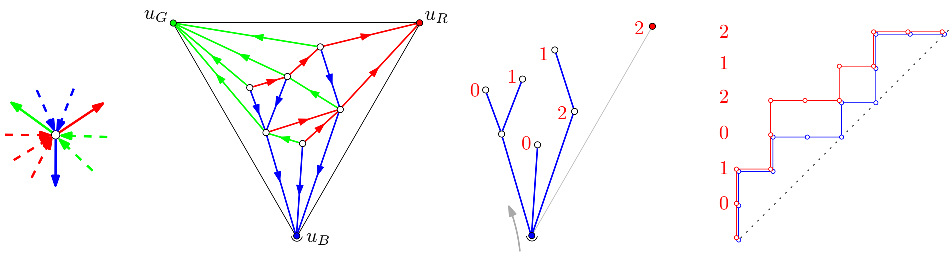

Figure 3. (a) Local rule for inner vertices in Schnyder woods. (b) A Schnyder wood with $n+3$ vertices ($n = 6$). (c) The blue tree with the indication of red indegrees at vertices. (d) The corresponding pair of Dyck walks in $\mathcal{P}_n$.

Let $\mathcal{P}_n$ be the set of pairs (γ, γ') of Dyck paths of length $2n$ such that γ' is above γ. The Bernardi-Bonichon construction [**2**] starts from a triangulation with $n+3$ vertices endowed with a Schnyder wood, and outputs a pair $(\gamma, \gamma') \in \mathcal{P}_n$. Precisely (see Figure 3 for an example), we let T_{blue} be the blue tree of the Schnyder wood plus the outer edge $\{u_B, u_R\}$, and let $v_0, \ldots, v_n = u_R$ be the vertices of $T_{\text{blue}} \backslash \{u_B\}$ ordered according to the first visit in a clockwise walk around T_{blue} starting at u_B. Then γ is obtained as the contour walk of $T_{\text{blue}} \backslash \{u_B, u_R\}$ and γ' is $NE^{\alpha_1} NE^{\alpha_2} \ldots NE^{\alpha_n}$, with α_r the number of ingoing red edges at v_r for $r \in [1..n]$. Bernardi and Bonichon show [**2**] that this gives a bijection between Schnyder woods on triangulations with $n+3$ vertices and $\mathcal{P}_n$; and they show that it specializes into a bijection between minimal Schnyder woods with $n+3$ vertices and $\mathcal{I}_n \subset \mathcal{P}_n$ (our Theorem 1 can be seen as an extension of this statement to separating decompositions).

On the other hand, minimal Schnyder woods are themselves known to be in bijection to certain tree structures [**3**]. A *3-mobile* is a (non-rooted) plane tree T where vertices have degree in $\{1, 3\}$ (those of degree 3 are called nodes, edges incident to leaves are called legs), the nodes are colored black or white so that adjacent nodes have different colors, all leaves are adjacent to black nodes, and the edges are colored blue, green or red such that around each node the incident edges in clockwise order are blue, green and red. Let $\mathcal{T}_n$ be the set of 3-mobiles with n white nodes. From a rooted triangulation M on $n+3$ vertices, endowed with its minimal Schnyder wood, one builds a 3-mobile $T \in \mathcal{T}_n$ as follows (see Figure 4): orient the outer cycle clockwise, insert a black vertex b_f in each inner face f of M, and then for each edge $e = u \to v$, with f, f' the faces on the left and on the right of e, create a new edge $\{u, b_f\}$ (if f is an inner face) and create a new pending edge (called a leg) at $b_{f'}$ pointing (but not reaching) to v; and finally erase the outer vertices and all the edges of M. Each edge of T gets the color of the corresponding corner of M.

Composing both constructions, we get a bijection between $\mathcal{I}_n$ and $\mathcal{T}_n$. Let $(\gamma, \gamma') \in \mathcal{I}_n$, with $n \geq 1$. For $(b, b') \in \{(0,0),(0,1),(1,1)\}$ we say that a position $r \in [0..n]$ is of type (b, b') if there is b (resp. b') at position r in $\mathrm{can}(\gamma)$ (resp. in $\mathrm{can}(\gamma'))$. Let $T \in \mathcal{T}_n$, with $n \geq 1$. A black node of T whose blue edge is a leg is said to be of type $(0,0)$ (resp. type $(1,1)$) if its red edge is a leg (resp. its green edge is a leg) and is said to be of type $(0,1)$ otherwise (only its blue edge is a leg).

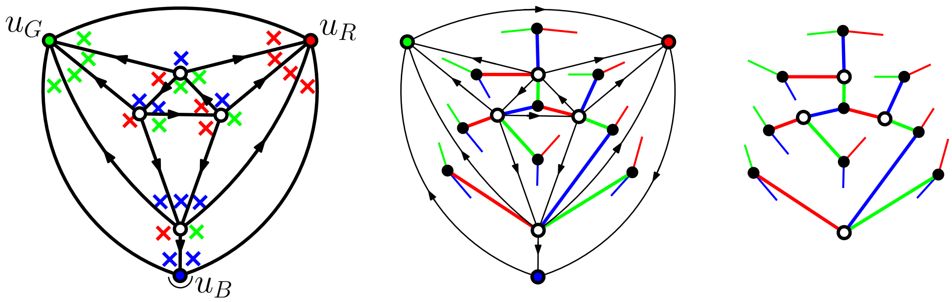

Figure 4. Left: A rooted triangulation endowed with its minimal Schnyder wood (colors are indicated at corners). Right: the corresponding 3-mobile.

Theorem 2. *Let $n \geq 1$. In the (composed) bijection between $\mathcal{I}_n$ and $\mathcal{T}_n$, for $(b, b') \in \{(0,0),(0,1),(1,1)\}$ each position of type (b, b') corresponds to a black node of type (b, b').*

Let $i, j, k \geq 0$, and $n = i + j + k + 1$. We denote by $a[i, j, k]$ the number of intervals in $\mathcal{I}_n$ having $i+1$ positions of type $(1,1)$, $j+1$ positions of type $(0,0)$ and k positions of type $(0,1)$, and we let $F(x, y, z) := \sum_{i,j,k} a[i, j, k] x^{i+1} y^{j+1} z^k$ be the associated generating function. Note that $F(x, y, 0) = \sum_{i,j} |\mathcal{S}_{i,j}| x^{i+1} y^{j+1} = \sum_{i,j} |\mathcal{G}_{i,j}| x^{i+1} y^{j+1}$.

Corollary 1. *The generating function $F \equiv F(x, y, z)$ is given by*

$$F = xR + yG + zRG - (x + zG)(y + zR)(1 + R)^2(1 + G)^2,$$

where R, G are the trivariate series (in x, y, z) specified by the system

$$\begin{cases} R &= (y + zR)(1 + R)(1 + G)^2, \\ G &= (x + zG)(1 + G)(1 + R)^2. \end{cases}$$

In particular $F(x, y, 0)$ coincides (upon setting $G = u/(1-u)$ and $R = v/(1-v)$) with the known expression [**6**, Eq. 2.3] of the bivariate series $\sum_{i,j} |\mathcal{Q}_{i,j}| x^{i+1} y^{j+1}$, and we recover $|\mathcal{G}_{i,j}| = |\mathcal{Q}_{i,j}|$; and $t + F(t,t,t)$ coincides (upon setting $G = R = \theta/(1-\theta)$) with the known expression [**19**, Eq. 4.9] of the series counting rooted simple triangulations by the number of vertices minus 2.

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