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TREE PIVOT-MINORS AND LINEAR RANK-WIDTH

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Abstract. Treewidth and its linear variant path-width play a central role for the graph minor relation. Rank-width and linear rank-width do the same for the graph pivot-minor relation. Robertson and Seymour (1983) proved that for every tree T there exists a constant c_T such that every graph of path-width at least c_T contains T as a minor. Motivated by this result, we examine whether for every tree T there exists a constant d_T such that every graph of linear rank-width at least d_T contains T as a pivot-minor. We show that this is false if T is not a caterpillar, but true if T is the claw.

1. Introduction

In order to increase our understanding of graph classes and their properties, it is natural to consider some notion of "width" and to research what properties a class of graphs whose width is bounded by a constant may have. In particular, this has been done in the context of graph containment problems, where the aim is to determine whether one graph appears as a "pattern" inside some other graph. Here, the definition of a pattern depends on the type of graph operations that we are allowed to use. For instance, a graph G contains a graph H as a *minor* if H can be obtained from G via a sequence of vertex deletions, edge deletions and edge contractions.

The notions of treewidth and its linear variant path-width are the most well-known width parameters. An important reason for this is their relevance in graph minor theory. In particular, Robertson and Seymour proved the following classical result.

Theorem 1.1 ([**22**]). *For every tree T, there exists a constant c_T such that every graph of path-width at least c_T contains T as a minor.*

We focus on the notion of linear rank-width, which can be seen as the linearisation of the notion of rank-width. The latter notion was introduced by Oum and Seymour [**21**] and expresses the minimum width k of a tree-like structure

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obtained by recursively splitting the vertex set of a graph in such a way that each cut induces a matrix of rank at most k (see Section 2 for a formal definition). Rank-width is more general than treewidth in the sense that every graph class of bounded treewidth has bounded rank-width, but there are classes for which the reverse does not hold [**7**]. The notion of rank-width has important algorithmic implications, as many NP-complete decision problems are known to be polynomial-time solvable not only for graph classes of bounded treewidth, but also for graph classes of bounded rank-width. Rank-width is equivalent to clique-width [**21**], another important and well-studied width parameter. Linear rank-width is equivalent to linear clique-width (see, for example, [**20**]) and is closely related to the trellis-width of linear codes [**14**].

The problem of determining whether a given graph has linear rank-width at most k for some given integer k is NP-complete (this follows from a result of Kashyap [**14**]). On the positive side, Jeong, Kim and Oum [**13**] gave an FPT algorithm for deciding whether a graph has linear rank-width at most k, whereas Ganian [**11**], and Adler, Farley and Proskurowski [**1**] characterised the graphs of linear rank-width at most 1.

To increase our understanding of rank-width and linear rank-width, we may want to verify if classical results such as Theorem 1.1 stay valid when we replace treewidth by rank-width and path-width by linear rank-width. However, it is known that edge deletions and contractions may increase the rank-width and linear rank-width [**6**]. This means that such graph operations are not useful for dealing with these parameters. Hence, instead of working with minors, Oum [**17**] proposed the notions of vertex-minors and pivot-minors, two closely related notions, which were called ℓ-reductions and p-reductions, respectively, in [**5**]. In this paper we focus on pivot-minors.

In order to define pivot-minors we need some additional terminology. The *local complementation* at a vertex u in a graph G replaces every edge of the subgraph induced by the neighbours of u by a non-edge, and vice versa. Let $G * u$ be the resulting graph. An *edge pivot* is the operation that takes an edge uv, first applies a local complementation at u, then at v, and then at u again. We denote the resulting graph $G \wedge uv = G*u*v*u$. As $G*u*v*u = G*v*u*v$, we observe that $G \wedge uv = G \wedge vu$. Alternatively, we can define the pivot of an edge uv as follows. Let S_u be the set of all neighbours of u non-adjacent to v, let S_v be the set of all neighbours of v non-adjacent to u and let S_{uv} be the set of common neighbours of u and v. First, we replace every edge between any two vertices in distinct sets from $\{S_u, S_v, S_{uv}\}$ by a non-edge and vice versa. Second, we delete every edge between u and S_u and add every edge between u and S_v, and similarly, delete every edge between v and S_v and add every edge between v and S_u. A graph G contains a graph H as a *pivot-minor* if H can be obtained from G by a sequence of vertex deletions and edge pivots.

Oum [**18**] showed that, for every positive constant k, the class of graphs of rank-width at most k is well-quasi-ordered under the pivot-minor relation. Kwon and Oum [**15**] proved that every graph of rank-width at most k is a pivot-minor of a graph of treewidth at most $2k$, and that a graph of linear rank-width at most k is a pivot-minor of a graph of path-width at most $k+1$. Geelen and Oum [**12**] characterized circle graphs in terms of forbidden pivot-minors. Oum [**19**] conjectured that for each fixed bipartite circle graph H, every graph G of sufficiently large rank-width contains H as a pivot-minor. In our previous paper [**9**], we proved that deciding whether a given graph G contains a given graph H as a pivot-minor is NP-complete, and we initiated a systematic study into the complexity of this problem when H is fixed and only G is part of the input.

In this paper we focus on the question of whether there exists an analogue to Theorem 1.1 for linear rank-width in terms of pivot-minors. Our first result provides a negative answer to this question. Here, a *caterpillar* is a tree that contains a path P, such that every vertex not on P has a neighbour on P. We prove the following.

Theorem 1.2. *If T is a tree that is not a caterpillar, then the class of T-pivot-minor-free graphs has unbounded linear rank-width.*

Due to Theorem 1.2, we may replace "tree" by "caterpillar" in our research question.

Question 1. Is it true that for every caterpillar T, there exists a constant d_T such that every graph of linear rank-width at least c_T contains T as a pivot-minor?

Question 1 turns out to be a challenging question, which remains largely unresolved. However, we have an affirmative answer if T is the *claw* (the 4-vertex star).

Theorem 1.3. *Every claw-pivot-minor-free graph has linear rank-width at most* 141.

2. Linear rank-width

Let G be a graph. Let A_G denote the *adjacency matrix* of G over the binary field. The *cut-rank function* of G is the function $\text{cutrk}_G\colon 2^{V(G)} \to \mathbb{N}_0$ such that for each $X \subseteq V(G)$,

$$\text{cutrk}_G(X)\colon = \text{rank}(A_G[X, V(G) \setminus X]),$$

where we compute the rank over the binary field. An ordering $(x_1, \ldots, x_n)$ of the vertex set $V(G)$ is called a *linear ordering* of G. The *width* of a linear ordering $(x_1, \ldots, x_n)$ of G is defined as $\max_{1\le i\le n}\{\text{cutrk}_G(\{x_1, \ldots, x_i\})\}$. The *linear rank-width* of G, denoted by $\text{lrw}(G)$, is defined as the minimum width over all linear orderings of G.

3. Proof sketch of Theorem 1.2

We introduce a class $\mathcal{C}$ of graphs containing graphs that look like the graph H in Figure 1. We show that $\mathcal{C}$ has unbounded linear rank-width and also that any tree that is not a caterpillar cannot be a pivot-minor of a graph in $\mathcal{C}$. Formally, we define $\mathcal{C}$ as the set of graphs that can be obtained from a tree T by subdividing each edge exactly once, and then taking a local complementation on every vertex

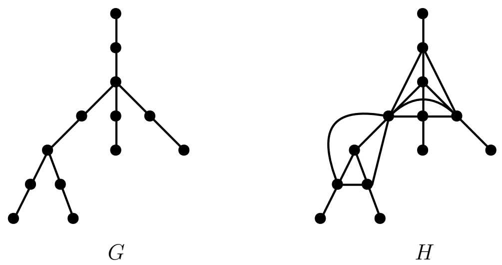

Figure 1. The construction of a graph H in $\mathcal{C}$.

that was originally included in T. We need to show two statements: (i) $\mathcal{C}$ has unbounded linear rank-width, and (ii) any tree that is not a caterpillar cannot be a pivot-minor of a graph in $\mathcal{C}$. Statement (i) can be deduced from the known facts that trees have unbounded linear rank-width due to Adler and Kanté [**2**], and that local complementations preserve linear rank-width. To prove (ii), we need a more involved argument using the notion of split decompositions.

Split decompositions, introduced by Cunningham [**8**], provide a tree-like structure of a graph with respect to its splits. A vertex subset A is *complete* to a vertex subset B if every vertex in A is adjacent to all vertices in B. A *split* (X, Y) in a graph G is a partition of $V(G)$ such that $|X|, |Y| \geq 2$ and the neighbourhood $N_G(X) \cap Y$ of X in Y is complete to the neighbourhood $N_G(Y) \cap X$ of Y in X. If a graph G admits a split (X, Y), we construct a new graph D on the vertex set $V(G) \cup \{x_1, y_1\}$ for some new vertices x_1 and y_1 such that (1) for vertices x, y with $\{x, y\} \subseteq X$ or $\{x, y\} \subseteq Y$, $xy \in E(G)$ if and only if $xy \in E(D)$, (2) x_1y_1 is a new edge called a *marked* edge, and no vertex in X has a neighbour in Y, (3) x_1 is adjacent to every vertex of $N_G(Y) \cap X$ and y_1 is adjacent to every vertex in $N_G(X) \cap Y$.

The marked edge x_1y_1 in D represents the fact that we decompose along a split given by $D - \{x_1, y_1\}$ in G. This graph D is called a *simple decomposition* of G. A *split decomposition* of a connected graph G is a graph D defined inductively to be either G or a graph obtained from a split decomposition D' of G by replacing a bag of D' with its simple decomposition, where a bag of D' is a connected component obtained by removing all marked edges. See Figure 2 for an example.

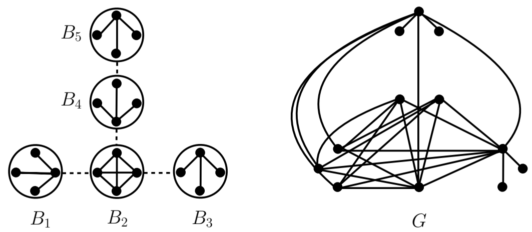

Figure 2. An example of a split decomposition of a graph G. Marked edges are dashed and each B_i is a bag.

Bouchet [**4**] investigated how a split decomposition can be modified when applying a local complementation at a vertex in the original graph. What is more important for us is that edge pivots can also be realised using local complementations. It is well known that for an edge uv in a graph, there is a unique path from u to v in its split decomposition such that this path starts and ends with an unmarked edge, and unmarked edges and marked edges appear alternately. Now, to obtain a split decomposition of a graph $G \wedge uv$, it is sufficient to apply edge pivots on unmarked edges on this path in each bag; see [**3**, Section 2.2] for detailed arguments. We mainly use the two observations that a bag that is a complete graph does not change when pivoting an edge, and a bag that is a star remains a star after pivoting an edge.

Now, graphs in our class $\mathcal{C}$ have the following type of split decompositions: the underlying decomposition tree is a subdivision of some tree, where each bag incident with at least three marked edges is a complete graph and every other bag is a star with exactly three vertices, whose centre is not incident with a marked edge. Therefore, any edge pivot can only change the shape of a star bag. And, if we remove a (real) vertex in some star bag, either the two components after removing this bag are disconnected, or these two decompositions are merged in such a way that neighbouring complete bags are merged into one complete bag. By this observation, we can deduce that every pivot-minor of a graph in $\mathcal{C}$ also has a split decomposition where each bag incident with at least three marked edges is a complete graph, and any other bag is a star graph or a complete graph (this is possible by removing all vertices of one part except one real vertex).

One subdivision of $K_{1,3}$ is the unique tree obstruction for being a caterpillar. But its split decomposition has a star bag incident with at least three marked edges. This means that it cannot be obtained as a pivot-minor of any graph in $\mathcal{C}$. This concludes our proof sketch of Theorem 1.2.

4. Proof sketch of Theorem 1.3

We may observe that every connected claw-pivot-minor-free graph is $(3P_1, W_4)$-free. Therefore, it is sufficient to concentrate on $(3P_1, W_4)$-free graphs. We can further show that taking the complement of a graph may increase its linear rank-width by at most 1. Since K_3 and $P_1 + 2P_2$ are the complements of $3P_1$ and W_4, respectively, it is sufficient to show that $(*)$ every connected $(K_3, P_1 + 2P_2)$-free graph has linear rank-width at most 140.

Let G be a $(K_3, P_1 + 2P_2)$-free graph. We prove that if we delete all but one vertex from each set of pairwise false twins, this does not decrease the linear rank-width of G by more than 1. So, we may assume that G has no false twins. The proof of $(*)$ consists of three parts: (1) Every connected bipartite $(P_1 + 2P_2)$-free graph containing no false twins has linear rank-width at most 4. (2) Every connected non-bipartite $(K_3, C_5, P_1 + 2P_2)$-free graph containing no false twins has linear rank-width at most 3. (3) Every connected $(K_3, P_1 + 2P_2)$-free graph containing C_5 and no false twins has linear rank-width at most 139.

Since a given graph has no false twins, the proof of (1) follows from Lozin [**16**], and (2) follows from [**10**]. We note that these results are proven for clique-width (and for rank-width). However, after some careful reformulations, we can prove that they also hold for *linear* rank-width. In particular, the proof for part (3) is somewhat different. We prove the following statement ($**$). Let G be a graph with partition (V_1, V_2, V_3) such that each V_i is independent, for every $a \in V_1, b \in V_2, c \in V_3$, the set $\{a, b, c\}$ is neither a clique nor an independent set, and all of $G[V_1 \cup V_2], G[V_2 \cup V_3], G[V_3 \cup V_1]$ are $2P_2$-free (such graphs are also known as bipartite chain graphs). Then G has linear rank-width at most 9. Briefly speaking, $2P_2$-free bipartite graphs admit a natural ordering of vertices, and because of the second assumption, the orderings of V_1, V_2, V_3 have to be "compatible". Thus, we can explicitly give an ordering of the vertices in such a way that complications only occur in the small layer.

Starting from a C_5, we classify all the vertices with respect to their neighbours on the C_5. The *bipartite complementation* takes two disjoint subsets A and B, and flips the adjacency relations between A and B. It is known that a bipartite complementation may change the linear rank-width by at most 2. We prove that by applying bipartite complementations at most 65 times, we can make the graph into a graph each of whose components is a 3-partite graph with the conditions in ($**$). In this way we can show that the original connected graph (without false twins) has linear rank-width at most 139.

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