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# A TURÁN-TYPE THEOREM FOR LARGE-DISTANCE GRAPHS IN EUCLIDEAN SPACES, AND RELATED ISODIAMETRIC PROBLEMS

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Abstract. A *large-distance graph* is a measurable graph whose vertex set is a measurable subset of $\mathbb{R}^d$, and two vertices are connected by an edge if and only if their distance is larger that 2. We address questions from extremal graph theory in the setting of large-distance graphs, focusing in particular on upper-bounds on the measures of vertices and edges of $K_r$-free large-distance graphs. Our main result states that if $A \subset \mathbb{R}^2$ is a measurable set such that the large-distance graph on $A$ does not contain any complete subgraph on three vertices then the 2-dimensional Lebesgue measure of $A$ is at most $2\pi$.

The results presented in this extended abstract are motivated by the following classical question from extremal graph theory. Let $k \geq 3$ be a given natural number. If we know the number $|V|$ of vertices in a simple graph $G = (V, E)$ which does not contain a complete subgraph on $k$ vertices $G = (V, E)$, what can we say about the relationship between the number $|E|$ of edges in $G$ and the number of vertices in $G$? The answer is provided by Turán's folklore result.

**Theorem 1** (Turán [**8**])**.** *Let $k \geq 3$ be a fixed natural number. Let $G = (V, E)$ be a graph which does not contain any complete subgraph on $k$ vertices. Then*

$$|E| \leq \frac{1}{2}\left(1 - \frac{1}{k-1}\right) \cdot |V|^2.$$

We shall be interested in a measure-theoretic counterpart of Turán's theorem. Instead of simple graphs, we consider the so called *large-distance graphs* whose underlying set is a subset of the Euclidean space $\mathbb{R}^d$ for some natural number $d$, and where two vertices are connected by an edge whenever they are far apart. This is covered by the following definition.

**Definition 1** (Large-distance graphs)**.** Let $A$ be a measurable subset of $\mathbb{R}^d$. Then we define the *large-distance graph* $\mathcal{G}_A$ corresponding to $A$ as follows: The

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vertex set of $\mathcal{G}_A$ is the set $A$. The edge set $\mathcal{E}_A$ of $\mathcal{G}_A$ is defined by

$$\mathcal{E}_A = \{(p, q) \in A \times A : \|p - q\| > 2\}.$$

Let us emphasize that the precise value of the distance threshold (which is set to 2 in the definition above) is not important. Only a very simple rescaling would be needed in order to reformulate all of our results for any other choice of the distance threshold.

Graphs (finite, or infinite such as here) with edges defined by similar metric conditions arise naturally in many real-life scenarios. For example, a finite version of large-distance graphs could be used for planning new train lines. Vertices would be cities and edges would then represent distant pairs of cities, where a high-speed train line would be desirable. A lot of research has been done on so-called *distance graphs* where two points of a subset of $\mathbb{R}^d$ are connected by an edge if and only if their distance equals 1, see e.g. [**7**]. Recall also that considering the $(d-1)$-dimensional unit sphere $S^{d-1}$ in $\mathbb{R}^d$ instead of an arbitrary subset $A$ of $\mathbb{R}^d$ and changing the distance threshold in Definition 1 to some $\alpha < 2$, leads to the so called *Borsuk graph* (see [**5**, p. 30], or [**4**]).

The main object of our study in large-distance graphs is introduced in the following.

**Definition 2** ($H$-free large-distance graphs)**.** Suppose that $A \subset \mathbb{R}^d$ is measurable and let $H = (V(H), E(H))$ be a finite, simple, graph on $k \geq 2$ labeled vertices, whose labels are represented by the set $\{1, \ldots, k\}$. Let $\mathcal{G}_A$ be the large-distance graph corresponding to $A$, and let $\mathcal{G}_A\langle H\rangle \subset A^k$ be the set consisting of all $k$-tuples, $(p_1, \ldots, p_k)$, of points in $A$ for which $(p_i, p_j) \in \mathcal{E}_A$ whenever $ij \in E(H)$. We say that $\mathcal{G}_A$ is *$H$-free* if $\mathcal{G}_A\langle H\rangle = \emptyset$.

Similarly to the motivating question from extremal graph theory mentioned above, we ask the following. Let $k \geq 3$ and $d \geq 2$ be given natural numbers. If we are given a set $A \subset \mathbb{R}^d$ for which $\mathcal{G}_A$ is $K_k$-free, what can we say about the relationship between the $2d$-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ and the $d$-dimensional Lebesgue measure of $A$? The answer is provided by a result of Bollobás.

**Theorem 2** (Bollobás [**1**])**.** *Let $k \geq 3$ be a given natural number. Let $A \subset \mathbb{R}^d$ be a measurable set for which $\mathcal{G}_A$ is $K_k$-free. Then the $2d$-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ is at most $\left(1 - \frac{1}{k-1}\right) \cdot \lambda_d(A)^2$.*

We provide a short proof of Theorem 2 using ideas from the theory of graphons which additionally allows us to characterize those sets for which the inequality reduces to an equality.

Now, given Theorem 2, we ask for absolute bounds on the "size" of the vertex-set and edge-set of a $K_k$-free large-distance graph. We refer to this problem as the Clique-isodiametric problem.

**Problem 3** (Clique-isodiametric problem)**.** Suppose that we are given $k, d \in \mathbb{N}$. Find

$$\mathfrak{V}_{d,k} = \sup_A \lambda_d(A) \quad \text{and} \quad \mathfrak{E}_{d,k} = \sup_A e(\mathcal{G}_A),$$

where the suprema range over all measurable sets $A \subset \mathbb{R}^d$ for which $\mathcal{G}_A$ is $K_k$-free.

Note that, while the problem in the finite setting is nontrivial only for $k \geq 3$, Problem 3 is interesting even in the easiest case when $k = 2$. Indeed, the solution of this special case is very well known as the *isodiametric inequality* (see [**3**, Theorem 2.4] or [**2**, Theorem 11.2.1]):

**Theorem 4** (Isodiametric inequality). *Let $A \subset \mathbb{R}^d$ be a measurable set and let* $\mathrm{diam}(A)$ *denote its diameter. Then*

$$\lambda_d(A) \leq \left(\frac{\mathrm{diam}(A)}{2}\right)^d \omega_d \ ,$$

*where $\omega_d$ denotes the $d$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^d$.*

By the isodiametric inequality, it immediately follows that for $k = 2$, the optimal upper bound for the $d$-dimensional Lebesgue measure of the set $A$ from Problem 3 equals $\omega_d$.

Unfortunately, we are not able to find a general solution to Problem 3, so we treat only the simplest case of $d = 2$ and $k = 3$. It turns out that the answer to the problem is very natural and predictable: the optimal upper bound is attained by the disjoint union of two unit balls that are far apart. However, the proof is very far from being trivial and contains some tedious computations. A precise formulation of the result follows.

**Theorem 5.** *Let $A \subset \mathbb{R}^2$ be a measurable set such that the corresponding large-distance graph $\mathcal{G}_A$ is triangle-free. Then the 2-dimensional Lebesgue measure of $A$ is at most $2\pi$.*

To prove this theorem, we distinguish three cases depending on the diameter $\mathrm{diam}(A)$ of the set $A$. The cases where either $\mathrm{diam}(A) \leq 2\sqrt{2}$ or $\mathrm{diam}(A) \geq 4$ are easy. In the last case where $2\sqrt{2} < \mathrm{diam}(A) < 4$ we proceed as follows. Suppose, without loss of generality, that $A$ is compact and find two points $p, q \in A$ whose distance realizes the diameter of $A$. Then $A$ is the disjoint union of the following three subsets: the set of points from $A$ that are in a distance $> 2$ from $p$, the set of points from $A$ that are in a distance $> 2$ from $q$, and the intersection of $A$ and of the two balls centered in $p$ and $q$, respectively, with radius 2. The measure of the intersection of the two balls centered in $p$ and $q$, respectively, with radius 2 can be easily computed (it depends on $\mathrm{diam}(A)$, of course). The most difficult part is to obtain upper bounds on the measure of the set of those points from $A$ that are in a distance $> 2$ from $p$ (or from $q$). To this end, we use the observation that the diameter of such a set is at most 2, and that such a set is contained in an annulus with inner radius 2 and outer radius $\mathrm{diam}(A)$. Then we prove an analogous result to the isodiametric inequality but with the additional assumption that the set under consideration is contained in the annulus. Finally, summing all the obtained upper bounds yields the result. Although we believe that some ideas from our proof could be useful even in the cases $k > 3$ or $d > 2$, we were not able to straightforwardly adapt our argument to this more general setting.

By combining Theorems 2 and 5, we obtain the following result.

**Theorem 6.** *Let $A \subset \mathbb{R}^2$ be a measurable set such that the large-distance graph $\mathcal{G}_A$ does not contain any complete subgraph on three vertices. Then the 4-dimensional Lebesgue measure of the edge set $\mathcal{E}_A$ is at most $2\pi^2$.*

As already mentioned, the proof of Theorem 5 is based on an isodiametric inequality on the annulus that might be interesting on its own. The precise formulation follows.

**Theorem 7** (Isodiametric inequality on the annulus)**.** *Let $R \in [2\sqrt{2}, 4]$ and consider the set $D := D(0,R) \setminus D(0,2)^o \subset \mathbb{R}^2$. Assume that $A$ is a measurable subset of $D$ such that $\operatorname{diam}(A) \le 2$. Then*

$$\lambda_2(A) \le R^2 \arcsin\left(\frac{\boldsymbol{a}}{R}\right) - 4\arcsin\left(\frac{\boldsymbol{a}}{2}\right) + 2\arccos(\boldsymbol{a}),$$

*where* $\boldsymbol{a} = \sqrt{\frac{-R^4+16R^2}{8(R^2+2)}}$.

The proof of Theorem 7 employs, among other things, Pólya's circular symmetrisation (see [**6**]) and allows to transform the set into a "well-behaved" and "maximal" subset of the annulus, $D$, whose diameter is less than or equal to 2 and whose measure is easy to compute.

Let us end with some remarks. For $d \le 4$, we conjecture that a measurable subset $A \subset \mathbb{R}^d$ for which $\mathcal{G}_A$ is $K_k$-free satisfies $\lambda_d(A) \le (k-1)\cdot\omega_d$. In other words, the extremal set of the clique-isodiametric problem is the union of $k-1$ many unit balls that are at sufficiently large distance apart. Let us remark that for $k = 3$ and $d \ge 5$, the optimal upper bound in Problem 3 is, maybe a little bit surprisingly, strictly larger than the volume of the union of two disjoint unit balls. Indeed, first notice that the radius of a 2-ball that circumscribes an equilateral triangle whose sides are equal to 2 is equal to $2/\sqrt{3}$. Now it is easy to see that any $d$-ball of radius $2/\sqrt{3}$ (for arbitrary $d \ge 2$) is a set for which the corresponding large-distance graph is $K_3$-free. The volume of such a $d$-ball is equal to $\left(2/\sqrt{3}\right)^d \cdot \omega_d$. Now it is not difficult to verify that $2\omega_d < \left(2/\sqrt{3}\right)^d \cdot \omega_d$, when $d \ge 5$, and therefore the optimal triangle-free set is not a disjoint union of two unit balls that are at sufficiently large distance. We do not have a conjecture for the optimal set in higher dimensions.

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