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GUARDING ISOMETRIC SUBGRAPHS AND LAZY COPS AND ROBBERS

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Abstract. In the game of Cops and Robbers, one of the most useful results is that an isometric path in a graph can be guarded by one cop. In this paper, we introduce the concept of wide shadow on a graph, and use it to provide a short proof of the characterization of 1-guardable graphs. As an application, we show that 3 cops can capture a robber in any planar graph with the added restriction that at most two cops can move simultaneously, proving a conjecture of Yang and strenghtening a classical result by Aigner and Fromme.

1. Introduction

The game of *cops and robbers* on graphs was introduced by Nowakowski and Winkler [**9**] and Quillot [**11**]. The game is played on a graph G by two players, the cop and robber. At the beggining, the cop chooses a vertex as his starting position, and after that the robber chooses his initial position. The players then move in alternate turns, and in each turn a player might stay at their current position, or move to a neighbour of their current position. The *cop wins* if he eventually occupies the same vertex as the robber, a situation we will refer to as *capturing the robber*, while the *robber wins* if he is able to indefinitely prevent this from happening. A graph where the cop has a winning strategy is called a *cop-win graph*.

The game was generalized by Aigner and Fromme in [**1**] to allow more than a single cop to play, and they defined the *cop-number* of a graph G, which we will denote by $c(G)$, to be the smallest integer k such that k cops can guarantee the robber's capture on G regardless of his strategy. Since for any graph G we have $c(G) \leq |V(G)|$, the cop-number is well defined for every finite graph. Moreover, for any graph G we have $c(G) \leq \gamma(G)$, where $\gamma(G)$ denotes the domination number of G.

A lot of research has been done studying the connections between a graph's topological properties and its cop-number (see [**4**] for a survey and [**5**] for more

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insight). The first result of this type is due to Aigner and Fromme [**1**], who showed that $c(G) \leq 3$ for any connected planar graph G. The main tool they used was a lemma showing that a cop can *guard* an isometric path in the graph.

Due to the importance and usefulness of this result, there has been interest in extending it to larger classes of graphs. Recenty, it was generalized to isometric trees in [**3**] and to "vertebrate graphs", a family of graphs including trees, in [**8**]. Our Theorem 2.2 generalizes this results and provides a characterization of the graphs that can be guarded by a cop whenever they appear as isometric subgraphs of a larger graph.

In [**10**], Offner and Ojakian introduced a variation of the game where only one cop is allowed to move at each turn, and refered to it as the *one-active-cop* game. Shortly after, this variation was introduced with different names, like *lazy cops and robber* [**2**], and the *one-cop-moves game* [**6**]. In this paper we follow the naming in [**6**], and define the k-*cops-move* number of a graph, $c_k(G)$, as the smallest integer such that $c_k(G)$ cops guarantee the robber's capture in G with the restriction that at most k cops can change their position each turn.

Sullivan et al. [**12**] showed that every graph G on at most eight vertices satisfies $c_1(G) \leq 2$, and that there is a unique graph on nine vertices with $c_1(G) = 3$. For several classes of planar graphs we have $c_1(G) = c(G)$, so they posed the question of whether there exists a planar graph G with $c_1(G) \geq 4$. Gao and Yang constructed a planar graph for which $c_1(G) \geq 4$ in [**6**], and Yang [**13**] conjectured that $c_2(G) \leq 3$ for any planar graph.

We introduce the concept of *wide shadow* and use it to obtain a short self-contained proof of Theorem 2.2, as well as to prove that $c_2(G) \leq 3$ for any planar graph, verifying Yang's conjecture.

All graphs in this paper are connected unless stated otherwise. Let G be a graph and $X, Y \subseteq V(G)$. We will use $d(X, Y)$ to denote the length of a shortest (X, Y)-path in G. In the case X has one vertex, we will write $d(x, Y)$ instead of $d(\{x\}, Y)$. The analogous will be done when Y or both sets consist of a single vertex.

2. Guarding isometric subgraphs using wide shadows

Let G be a graph, and let H be an isometric subgraph of G. For $v \in V(G)$ and $x \in V(H)$, let $H_v(x) = \{y \in V(H) : d(x, y) \leq d(x, v)\}$. We define the *wide shadow* of v on H to be the set $S_H(v) = \bigcap_{x \in V(H)} H_v(x)$.

In general, the wide shadow of a vertex on an isometric subgraph H may be empty, but this is not the case when H is a Helly graph.

A family of sets $\mathcal{S}$ has the *Helly property* if for every $\mathcal{T} \subseteq \mathcal{S}$ we have the following property: if $X_1 \cap X_2 \neq \emptyset$ for every $X_1, X_2 \in \mathcal{T}$, then $\cap \mathcal{T} \neq \emptyset$. Let $N^k[u] = \{v \in V(G) : d(v, y) \leq k\}$. A graph is a *Helly graph* if the family $\{N^k[u] : u \in V(G), k \geq 0\}$ has the Helly property. In particular, trees are Helly graphs.

Lemma 2.1. *Let G be a graph and H an isometric subgraph of G. If H is a Helly graph, then:*

(i) *for every* $v \in V(G)$, $S_H(v) \neq \emptyset$;
(ii) *for every* $uv \in E(G)$, *and every* $x \in S_H(u)$, *we have* $d(x, S_H(v)) \leq 1$.

Proof. Part (i) follows from the definition of Helly Graph and the triangle inequality. For part (ii), let $uv \in E(G)$. Notice that for every $x \in V(H)$, we have $d(x,u) - 1 \leq d(x,v) \leq d(x,u) + 1$. This implies $H_v(x) \cap N_H[y] \neq \emptyset$ for every $x \in V(H)$ and $y \in S_H(u)$, so we have $S_H(v) \cap N_H[y] \neq \emptyset$ for every $y \in S_H(u)$. It follows that every vertex $y \in S_H(u)$ either $y \in S_H(v)$ or there exists $x \in N_H(y) \cap S_H(v)$, hence $d(y, S_H(v)) \leq 1$, completing the proof. □

Let G be a graph, H a subgraph of G and k a positive integer. We say that H is *k-guardable in G* if, after finitely many moves, k cops can move on vertices of H in such a way that, if the robber enters H, then he will be captured in the next turn. The high-level idea is that one can k-guard a subgraph H if there are k cops in vertices of H and, from their positions, every vertex in H can be reached by a cop "at least as quickly as the robber". Using the concept of wide shadow we are able to characterize the class of 1-guardable graphs.

Theorem 2.2. *A graph H is* 1*-guardable if and only if H is a Helly graph.*

In order to prove Theorem 2.4, we need the concept of a bypath. Let G be a graph, H a subgraph of G, and $P = v_1v_2\dots v_k$ an isometric path in H. A path $B = b_1b_2\dots b_t$ ($t \geq 3$) contained in H is called a *bypath* of P in H if $B \cap P = \{b_1, b_t\}$, and the path $P_{\langle B\rangle} = Pb_1Bb_tP$ is also an isometric path in H. A path P is *bypath-free* in H if H contains no bypath of P.

We say that an induced subgraph H is *k-leisurely-guardable* if it is k-guardable, and there exists a k-guarding strategy of H for a set of cops $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ such that, after a finite number of moves, there is a turn where either the robber enters H or at least one C_i can stay still and H is still being k-guarded by $\mathcal{C}$. In the case of paths, we can find the conditions under which they can be 1-leisurely guarded.

Lemma 2.3. *Let G be a planar graph and P an isometric path in G. If P is bypath-free in G, then P is* 1*-leisurely-guardable.*

Proof. After a finite number of moves, we can get one cop C to move to a vertex in the wide shadow of the robber on P. Once C is in $S_P(R)$, she will stay still if she is in $S_P(R)$ after the robber's turn, and will move when her position after the robber's turn is not in the wide shadow of the robber. By Lemma 2.1, the cop can always get back in the wide shadow of R with a single move. By staying in the wide shadow of the robber, the cop can 1-guard P.

If P has length ℓ, with $\ell \leq 2$, the cop can guard it without moving by simply staying at some vertex of P, so we may assume $\ell \geq 3$. Since P is bypath-free in G, we have $|S_P(R)| \geq 2$, so the robber can only move at most ℓ consecutive times without entering P and forcing the cop to move to stay in his wide shadow. Hence, after at most ℓ turns, the robber will either enter P or the cop can stay still and be 1-guarding P, so P is 1-leisurely-guardable. □

Theorem 2.4. $c_2(G) \leq 3$ *for every planar graph G.*

Sketch of Proof. The proof follows the idea of moving the cops in such way that, after a finite number of turns, the set of vertices which the robber can enter without being captured, which we call the *robber's territory*, is reduced.

1. Place three cops on one vertex v.
2. Move one cop to guard a path P in the robber's territory which is isometric and bypath-free. Since P is bypath-free, the cop can leisurely-guard the path.
3. Move a second cop to guard a path Q that has initial and terminal vertices are in P, at least one intermediate vertex in the robber's territory, and which is isometric in the robber's territory.
4. Since P is being leisurely-guarded, we can use the turns when that cop stays put to move the third cop to a path R whose initial and terminal vertices are in P and Q with at least one vertex in the robber's territory. The path R should be chosen so that, the first turn the cops are guarding the three paths, the the robber's territory has been restricted in such way that at least one of the paths surrounding the robber's territory is being leisurely guarded.
5. Release the cop which is guarding a path not adjacent to the robber's territory.
6. Repeat from 4 until the robber's territory is empty.

□

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