Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 779–785

ON SOME EXTREMAL RESULTS FOR ORDER TYPES

J. HAN, Y. KOHAYAKAWA, M. T. SALES and H. STAGNI

Abstract. A *configuration* is a finite set of points in the plane. Two configurations A and B have the same *order type* if there exists a bijection between them preserving the orientation of every ordered triple. We investigate the following extremal problem on embedding configurations in general position in integer grid. Given an order type B, let $\mathrm{ex}(N, B)$ be the maximum integer m such that there exists a subconfiguration of the integer grid $[N]^2$ of size m without a copy of B. An application of the celebrated multidimensional Szemerédi's theorem gives $\mathrm{ex}(N, B) = o(N^2)$.

We first prove a subquadratic upper bound for all large order types B and large N, namely, $\mathrm{ex}(N, B) \leq N^{2-\eta}$ for some $\eta = \eta(B) > 0$. Then we give improved bounds for specific order types: we show that $\mathrm{ex}(N, B) = O(N)$ for the convex order type B, and $\mathrm{ex}(N, B) = N^{3/2+o(1)}$ for those B satisfying the so-called Erdős-Hajnal property. Our approach is to study the inverse problem, that is, the smallest $N_0 = N_0(\alpha, B)$ such that every α proportion of $[N_0]^2$ contains a copy of B.

1. Introduction and main results

A *configuration* is a finite set of points on the plane. Given an ordered triple $(x, y, z) \in (\mathbb{R}^2)^3$, we define the *orientation* $\chi(x, y, z)$ by the sign of the area of xyz,

$$\chi(x, y, z) := \mathrm{sgn}[xyz] = \mathrm{sgn} \begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}.$$

Thus $\chi(x, y, z)$ is positive when (x, y, z) is in couterclockwise orientation and negative when (x, y, z) is in clockwise orientation. Two configurations A and B have the *same order type* or are *isomorphic* (denoted $A \cong B$) if there exists a bijection $\iota\colon A \to B$ such that $\chi(x, y, z) = \chi(\iota(x), \iota(y), \iota(z))$ for all $x, y, z \in A$. Moreover, a configuration A *contains* a copy of order type B if there is $B' \subseteq A$ such that $B' \cong B$. When there is no danger of confusion, we write $B \subseteq A$ to mean that

Received June 7, 2019.
2010 *Mathematics Subject Classification.* Primary 52C35, 52C45, 05D99.
Y.K. is partially supported by FAPESP (2013/03447-6) and CNPq (311412/2018-1, 423833/2018-9).The third author is partially supported by the NSF (DMS 1764385) and was supported by CNPq (130483/2016-8). H.S. is supported by FAPESP (2015/15986-4, 2017/02263-0).This study was financed in part by the Coordenaao de Aperfeioamento de Pessoal de Nível Superior, Brasil (CAPES), Finance Code 001.
FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

order type A contains a copy of order type B. For our purposes every order type is in general position, that is, they never contain three collinear points.

The notion of order types captures much of the combinatorics of point sets in the plane. For instance, if A and B are the vertices of a convex n-gon, then $A \cong B$, and, conversely, if A is such a *convex configuration* and $A \cong B$, then B is also convex. A more sophisticated example is the fact that the convex hull of an n-point configuration can be computed in $O(n \log n)$ time, solely by means of orientation queries and, in particular, it is determined by the order type of the given configuration [**16**]. For a beautiful overview of combinatorial, geometric and computation problems and results concerning order types, the reader is referred to the recent monograph of Eppstein [**4**]. Another inspiring source discussing classical problems and results involving order types that are related to the problems investigated here is [**2**, Chapter 8].

Given an order type B, let $\mathrm{gr}(B)$ be given by

$$\mathrm{gr}(B) = \min\{N \in \mathbb{N}\colon B \subseteq [N]^2\},$$

the minimum integer N such that the grid $[N]^2 = \{1, \dots, N\}^2 \subseteq \mathbb{Z}^2$ contains a copy of order type B. Since a class of equivalence of an order type in general position is an open set, we obtain, by approximating a configuration over the rational plane, that $\mathrm{gr}(B)$ always exists. Note that this is not the case if we drop the assumption of configurations in general position (see [**10**, p. 33], or [**4**, Section 13.1]).

The behavior of $\mathrm{gr}(B)$ has been studied since early last century. For instance, Jarník [**13**] showed that $\mathrm{gr}(B) = \Theta(n^{3/2})$ for convex B of size n and Bereg et al. [**1**] showed similar bounds for the 'double circle'. For a general order type, Goodman, Pollack and Sturmfels [**8**] proved that gr grows doubly exponentially for sufficiently large n, in the sense that there exists constants c_1 and $c_2 > 0$ such that if A is an order type of size n, then $\mathrm{gr}(A) \leq \exp(2^{c_2 n})$ and for every n there exists an order type B with $\mathrm{gr}(B) \geq \exp(2^{c_1 n})$.

Here, we are interested in the following extremal problem concerning order types in a grid. Let $\mathrm{ex}(N, B)$ be the maximum integer m such that there exists a subconfiguration of $[N]^2$ of size m without a copy of B. Clearly, by the previous paragraph, $\mathrm{ex}(N, B) = N^2$ for $N < \mathrm{gr}(B)$. The aim of this paper is to provide a general upper bound for $\mathrm{ex}(N, B)$ for large order types B and sufficiently large N.

Since order types are preserved by affine transformations, one possible attempt is to determine the maximum size of a configuration of $[N]^2$ without a homothetic translation of a configuration B. This can be done by the multidimensional Szemerédi's theorem [**7, 9, 17, 18**], which gives us $\mathrm{ex}(N, B) = o(N^2)$. Unfortunately, the bound provided by this approach is very weak. In order to see that, consider the following inverse problem: Let $N_0(B, \alpha)$ be the minimum integer N_0 such that if $N \geq N_0$, then every subset $X \subseteq [N]^2$ with $|X| \geq \alpha N^2$ contains a copy of B. A straightforward application of the multidimensional Szemerédi's theorem implies that $N_0(B, \alpha)$ exists, but with an Ackermann-type upper bound. Our first result provides a much better bound for $N_0(B, \alpha)$.

Theorem 1.1. *Let B be an order type of size n in general position and $0 < \alpha < 1$. Then*

$$N_0(B,\alpha) \le (3\,\mathrm{gr}(B))^{3n\log(1/\alpha)}.$$

Since $\mathrm{gr}(B)$ is no more than doubly exponential in n, we obtain that in the worst scenario $N_0(B,\alpha)$ is doubly exponential in n and $\log(1/\alpha)$. Also the upper bound provided by Theorem 1.1 gives us a bound on $\mathrm{ex}(N,B)$ of the form $N^{2-\eta}$, where $\eta = \eta(B) > 0$.

Corollary 1.2. *Let B be an order type of size n in general position. Then*

$$\mathrm{ex}(N,B) \le N^{2-\eta},$$

where $\eta := (3n\log(3\,\mathrm{gr}(B)))^{-1}$.

Proof. Write $g := 3\,\mathrm{gr}(B)$ and let $\alpha = N^{-(3n\log g)^{-1}}$. Thus, we have $N = g^{3n\log(1/\alpha)} \ge N_0(B,\alpha)$. By the definition of $N_0(B,\alpha)$ we have $\mathrm{ex}(N,B) \le \alpha N^2$. Thus the corollary follows. □

We next show that, for some specific order types, we can get significantly improved bounds. For convex configurations, we show that the extremal number is linear in the side-length of the grid.

Theorem 1.3. *Let B be the convex order type of size n. Then*

$$\mathrm{ex}(N,B) \le 2^{n+o(n)}N.$$

The famous Erdős-Szekeres problem asks for the maximum number $\mathrm{ES}(N)$ of points in general position which contain no copy of a convex N-gon. Erdős and Szekeres [**5**] showed that this number is between $2^{N-2} \le \mathrm{ES}(N) \le \binom{2N-4}{N-2} + 1$. A recent breakthrough due to Suk [**20**] gives $\mathrm{ES}(N) \le 2^{N+o(N)}$. Let $\mathrm{ES}(N,B)$ be the maximal size of a configuration in the plane with no copy of B and no copy of a convex N-gon. We say that an order type B satisfies the *Erdős-Hajnal property with exponent c* if $\mathrm{ES}(N,B) \le N^c$ for every sufficiently large N. That is, forbidding an extra order type B brings down the parameter $\mathrm{ES}(N)$ significantly. This concept was introduced by Károlyi, Solymosi and Toth [**14, 15**] to generalize the Erdős-Szekeres problem. They also give examples of order types that have the Erdős-Hajnal property. As another example, we [**11**] showed that B-freeness is efficiently testable for order types B that have the Erdős-Hajnal property.

We prove a stronger bound on $\mathrm{ex}(N,B)$ for order types B that satisfy the Erdős-Hajnal property.

Theorem 1.4. *Let the configuration B satisfy the Erdős-Hajnal property with exponent c. Then there exists N_0 such that for every $N \ge N_0$ we have*

$$\mathrm{ex}(N,B) \le 4N^{3/2}(\log N)^{2c+1}.$$

In Section 2, we present the main idea of our method and give the proof of Theorem 1.1. In Section 3, we prove Theorems 1.3 and 1.4. In what follows, we make no attempt to optimize the calculations in our proofs.

2. Main idea and proof of Theorem 1.1

2.1. A general approach for estimating $N_0(B,\alpha)$

Let B be an order type of size n in general position, $0 < \alpha \leq 1$ a real number and A a configuration, we say that $A \to_\alpha B$ if every α-proportion of A contains a copy of B. That is, every subset $A' \subseteq A$ with $|A'| \geq \alpha|A|$ contains a copy of B. Given B and α, the problem of finding A such that $A \to_\alpha B$ is strongly related to the parameter $N_0(B,\alpha)$. The reason is the following averaging lemma, which shows that a dense set on a grid contains a big proportion of *any* given configuration.

Lemma 2.1 ([**12**]). *Let N be an integer and A a configuration such that $A \subseteq [N]^2$. Then for every α-proportion X of $[N]^2$, there exists a translation A' of A such that X contains an $\alpha/4$-proportion of A', that is, $|X \cap A'| \geq \alpha|A'|/4$.*

Applying Lemma 2.1 with a configuration A such that $A \to_{\alpha/4} B$ and $N = \mathrm{gr}(A)$ we obtain the following corollary.

Corollary 2.2. *Let B be an order type and $\alpha > 0$. If A is an order type such that $A \to_{\alpha/4} B$, then $N_0(B,\alpha) \leq \mathrm{gr}(A)$.*

Thus, Corollary 2.2 gives us a general strategy for obtaining upper bounds for $N_0(B,\alpha)$. The strategy consists of two steps: given $\alpha > 0$ and B, we first construct a configuration A in the plane such that $A \to_\alpha B$, and then we estimate $\mathrm{gr}(A)$.

2.2. Iterative blow-ups and the proof of Theorem 1.1

We say that a configuration A is a *k-blow-up* of an n-point configuration B if there exists a partition $A = A_1 \cup \cdots \cup A_n$ with $|A_1| = \cdots = |A_n| = k$ such that every transversal is isomorphic to B, i.e., every set $X = \{x_1,\ldots,x_n\}$ with $x_i \in A_i$ is isomorphic to B. A configuration A is a *C-blow-up* of an n-point configuration B if A is a $|C|$-blow-up of B and every $A_i \cong C$.

Since a class of equivalence of an order type in general position is open, there is a simple way to construct blow-ups. Write $B := \{x_1,\ldots,x_n\}$ and let $B_\varepsilon(x_i)$ be the open ball of radius ε centered at x_i. By considering ε sufficiently small, one can notice that every transversal of $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)$ is isomorphic to B. Finally, we can easily find a copy of any order type inside each $B_\varepsilon(x_i)$, in particular C. Therefore we obtain a C-blow-up of B.

Another remark is that this construction is not unique and therefore there are more than one C-blow-ups of B. Because of this we let $B \otimes C$ be the set of all C-blow-ups of B. We extend this notation for more than two configurations. Let $X_1,\ldots,X_d$ be d configurations with $n_1,\ldots,n_d$ points, respectively. We define $\bigotimes_{i=1}^d X_i$ for $d \geq 2$ as the set of order types A such that there exists a partition $A = A_1 \ldots A_{n_1}$ with $A_j \in \bigotimes_{i=2}^d X_i$, $j \in [n_1]$ satisfying that every transversal is isomorphic to X_1.

For an order type B, the blow-up gives an example of configuration A such that $A \to_\alpha B$. Now we recall the following result from [**12**].

Theorem 2.3 ([**12**])**.** *Let B be a configuration in general position of size $n \geq 1$, $0 < \alpha < 1$ a real number and $d = \lceil n \log(1/\alpha) \rceil$. Then for every $A \in \bigotimes_{i=1}^{d} B$ we have that $A \to_\alpha B$.*

Recall that every configuration can be embedded in a grid of doubly exponential side-length. Therefore, Theorem 2.3 is enough to give an upper bound on $N_0(B, \alpha)$. However, one can improve the upper bound by constructing the blow-up manually inside the grid. This is done by the following lemma, proved in [**12**].

Corollary 2.4 ([**12**])**.** *Given an order type B in general position and d an integer, there exists a configuration $A \in \bigotimes_{i=1}^{d} B$ satisfying that* $\mathrm{gr}(A) \leq (3\,\mathrm{gr}(B))^{2d-1}$.

Now Theorem 1.1 follows immediately from these auxiliary results.

Proof of Theorem 1.1. Given any order type B of size $n \geq 1$ and $\alpha > 0$, let $d = \lceil n \log(1/\alpha) \rceil$. Take the configuration $A \in \bigotimes_{i=1}^{d} B$ given by Corollary 2.4 satisfying that $\mathrm{gr}(A) \leq (3\,\mathrm{gr}(B))^{2d-1}$. Since $A \to_\alpha B$ by Theorem 2.3, we obtain by Corollary 2.2 that

$$N_0(B, \alpha) \leq \mathrm{gr}(A) \leq (3\,\mathrm{gr}(B))^{2d-1} \leq (3\,\mathrm{gr}(B))^{3n \log(1/\alpha)}. \qquad \square$$

3. Specific order types

In Section 2 we discussed a general strategy to obtain upper bounds for $N_0(B, \alpha)$. The result can be significantly improved for certain order types. In this section we give improved bounds for two particular cases: B convex and B satisfying the Erdős-Hajnal property.

For B convex of size n, it turns out that A should be taken as a configuration in general position of size $2^{n+o(n)}/\alpha$. The reason is that any α-proportion of A contains $2^{n+o(n)}$ points and by the aforementioned result of Suk [**20**] it contains a convex configuration of size n. Thus $A \to_\alpha B$. To estimate $\mathrm{gr}(A)$ we use the following result of Erdős, which relates to a puzzle by Dudeney (1917)[1].

Theorem 3.1 ([**19**])**.** *There exists a configuration A_0 of m points in general position such that* $\mathrm{gr}(A) \leq 2m$.

Applying Corollary 2.2 to the configuration A_0 given by the theorem above, we obtain that $N_0(B, \alpha) \leq 2^{n+o(n)}/\alpha$, which implies Theorem 1.3.

We now turn to the proof of Theorem 1.4. Thus, let a configuration B satisfying the Erdős-Hajnal property be given. The idea is to construct a configuration A with no large convex subsets that, at the same time, have reasonably small $\mathrm{gr}(A)$. This can be achieved by the following result due to Duque, Fabila-Monroy and Hidalgo-Toscano [**6**] on the grid parameter of the Erdős-Szekeres lower bound construction.

Theorem 3.2 ([**3**])**.** *The Erdős-Szekeres construction of $n = 2^{t-2}$ points can be realized in an integer grid of side-length $O(n^2(\log n)^3)$.*

[1]The puzzle asks how to place 16 pawns on a chessboard, without allowing any lines of three pawns.

In the forthcoming full version of this paper we shall give another construction with a simpler proof that can be used in the proof of Theorem 1.4. Here, we give a proof of Theorem 1.4 based on Theorem 3.2.

Proof of Theorem 1.4. Let $c \geq 1$ be the constant such that $\mathrm{ES}(N, B) \leq N^c$ for every sufficiently large N. The Erdős-Szekeres construction is a configuration of size $n = 2^{t-2}$ with no convex sets of size t. Let A_1 be the configuration given by Theorem 3.2 with $t = \log((1/\alpha)(\log 1/\alpha)^{2c})$, where $\alpha = \alpha(N)$ will be chosen in a moment. Then for every sufficiently small α, every α-proportion of A_1 contains $\alpha 2^{t-2} = \frac{1}{4}(\log(1/\alpha))^{2c} > t^c \geq \mathrm{ES}(t, B)$ points. Therefore $A_1 \to_\alpha B$ and we conclude by Corollary 2.2 that

$$N_0(B, 4\alpha) \leq \mathrm{gr}(A_1) = n^2(\log n)^3 \leq t^3 2^{2t} \leq (1/\alpha)^2(\log(1/\alpha))^{4c+2},$$

since α is small enough. Let $\alpha = (\log N)^{2c+1}/N^{1/2}$. Then, we have

$$N_0(B, 4\alpha) \leq (1/\alpha)^2(\log(1/\alpha))^{4c+2} < \frac{N(\log(N^{1/2}))^{4c+2}}{(\log N)^{4c+2}} < N.$$

By the definition of $N_0(B, 4\alpha)$ we obtain $\mathrm{ex}(N, B) \leq 4N^{3/2}(\log N)^{2c+1}$. □

Acknowledgment. The authors are grateful to the referees for their comments and encouragement.

References

1. Bereg S., Fabila-Monroy R., Flores-Peñaloza D., Lopez M. A. and Pérez-Lantero P., *Embedding the double circle in a square grid of minimum size*, Internat. J. Comput. Geom. Appl. **24** (2014), 247–258.

2. Brass P., Moser W. and Pach J., *Research Problems in Discrete Geometry*, Springer, New York, 2005.

3. Duque F., Fabila-Monroy R. and Hidalgo-Toscano C., *Point sets with small integer coordinates and no large convex polygons*, Discrete Comput. Geom. **59** (2018), 461–476.

4. Eppstein D. *Forbidden Configurations in Discrete Geometry*, Cambridge University Press, 2018.

5. Erdős P. and Szekeres G., *A combinatorial problem in geometry*, Compos. Math. **2** (1935), 463–470.

6. Erdős P. and Szekeres G., *On some extremum problems in elementary geometry*, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. **3–4** (1961), 53–62.

7. Furstenberg H. and Katznelson Y., *An ergodic Szemerédi theorem for commuting transformations*, Journal d'Analyse Mathématique **34** (1978), 275–291.

8. Goodman J. E., Pollack R. and Sturmfels B., *Coordinate representation of order types requires exponential storage*, in: Proceedings of the Twenty-first Annual ACM Symposium on Theory of Computing, New York, USA, 1989, STOC '89, ACM, 405–410.

9. Gowers T., *Hypergraph regularity and the multidimensional Szemerédi theorem*, Ann. of Math. **166** (2007), 897–946.

10. Grünbaum B., Arrangements and Spreads, Amer. Math. Soc., Providence, 1972.

11. Han J., Kohayakawa Y., Sales M. T. and Stagni H., *Property testing for point sets on the plane*, in: LATIN 2018: Theoretical informatics, Lecture Notes in Comput. Sci. 10807, Springer, Cham, 2018, 584–596.

12. Han J., Kohayakawa Y., Sales M. T. and Stagni H., *Extremal and probabilistic results for order types*, in: Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, 2019, 426–435.

13. Jarník V., *Über die Gitterpunkte auf konvexen Kurven*, Mathematische Zeitschrift **24** (1926), 500–518.
14. Károlyi G. and Solymosi J., *Erdős-Szekeres theorem with forbidden order types*, J. Combin. Theory A **113** (2006), 455–465.
15. Károlyi G. and Toth G., *Erdős-Szekeres for point sets with forbidden subconfigurations*, Discrete Comput. Geom. **48** (2012), 441–452.
16. Knuth D. E., *Axioms and hulls*, Lecture Notes in Comput. Sci. 606, Springer-Verlag, 1992.
17. Nagle B., Rödl V. and Schacht M., *The counting lemma for regular k-uniform hypergraphs*, Random Structures Algorithms **28** (2006), 113–179.
18. Rödl V. and Skokan J., *Regularity lemma for k-uniform hypergraphs*, Random Structures Algorithms **25** (2004), 1–42.
19. Roth K. F., *On a problem of Heilbronn*, J. London Math. Soc. **26** (1951), 198–204.
20. Suk A., *On the Erdős-Szekeres convex polygon theorem*, J. Amer. Math. Soc. **30** (2017), 1047–1053.

J. Han, Department of Mathematics, University of Rhode Island, Kingston, USA,
e-mail: `jie_han@uri.edu`

Y. Kohayakawa, Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil,
e-mail: `yoshi@ime.usp.br`

M. T. Sales, Department of Mathematics, Emory University, Atlanta, USA,
e-mail: `marcelo.tadeu.sales@emory.edu`

H. Stagni, Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil,
e-mail: `stagni@ime.usp.br`