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SHARP BOUNDS FOR THE CHROMATIC NUMBER OF RANDOM KNESER GRAPHS

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ABSTRACT. Given positive integers $n \geq 2k$, a *Kneser graph* $KG_{n,k}$ is a graph whose vertex set is the collection of all k-element subsets of the set $\{1, \dots, n\}$, with edges connecting pairs of disjoint sets. A famous result due to L. Lovász states that the chromatic number of $KG_{n,k}$ is equal to $n - 2k + 2$. In this paper, we study the *random Kneser graph* $KG_{n,k}(p)$, obtained from $KG_{n,k}$ by including each of the edges of $KG_{n,k}$ independently and with probability p.

We prove that, for any fixed $k \geq 3$, $\chi(KG_{n,k}(1/2)) = n - \Theta(\sqrt[2k-2]{\log_2 n})$. We also provide new bounds for the case of growing k. This significantly improves previous results on the subject, obtained by Kupavskii and by Alishahi and Hajiabolhassan. We also discuss an interesting connection to an extremal problem on embeddability of complexes.

For positive integers n, k, where $n \geq 2k$, a *Kneser graph* $KG_{n,k} = (V, E)$ is a graph, whose vertex set V is the collection of all k-element subsets of the set $[n] := \{1, \dots, n\}$, and E is the collection of the pairs of disjoint sets from V. It was introduced by Kneser [**21**], who showed that $\chi(KG_{n,k}) \leq n - 2k + 2$. He conjectured that this inequality is in fact an equality. This was proven by Lovász [**25**], who introduced the use of topological methods in combinatorics in that paper.

We remark that independent sets in $KG_{n,k}$ are *intersecting families*, and it is a famous result of Erdős, Ko and Rado [**12**] that $\alpha(KG_{n,k}) = \binom{n-1}{k-1}$. For results on the independence sets of Kneser graphs and hypergraphs, see [**11, 13, 14, 15, 16, 18, 24, 34**].

The notion of the *random Kneser graph* $KG_{n,k}(p)$ was introduced in [**5, 6**]. For $0 < p < 1$, the graph $KG_{n,k}(p)$ is constructed by including each edge of $KG_{n,k}$ in $KG_{n,k}(p)$ independently with probability p. The authors of [**7**] studied the independence number of $KG_{n,k}(p)$. Later, their results were strengthened in [**3, 8, 9**]). Interestingly, the independence number of $KG_{n,k}(p)$ stays *exactly* the same as the independence number of $KG_{n,k}$ in many regimes. Independence numbers of random subgraphs of generalized Kneser graphs and related questions were studied in [**4, 5, 6, 10, 17, 27, 28, 29, 30, 31, 32, 33**]. In [**22**], the second author proposed to study the chromatic number of $KG_{n,k}(p)$. He proved that in different regimes the chromatic number of $KG_{n,k}(p)$ is very close to that of $KG_{n,k}$.

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In particular, he showed that, for any constant k and p, there exists a constant C, such that a.a.s.[1]

$$\chi(KG_{n,k}(p)) \geq n - Cn^{\frac{3}{2k}}. \tag{1}$$

A better a.a.s. bound was next obtained by Alishahi and Hajiabolhassan [**1**]. In a follow-up paper, the second author [**23**] improved the inequality (1) to

$$\chi(KG_{n,k}(p)) \geq n - C(n \log n)^{1/k} \tag{2}$$

for some $C = C(p, k)$. The main result of this paper is the following theorem, which, in particular, significantly improves on the bounds (1) and (2) and settles the problem in the case of constant k.

Theorem 1. *For any fixed $0 < p < 1$, $k \geq 3$ and $n \to \infty$, we a.a.s. have*

$$\chi(KG_{n,k}(p)) = n - \Theta\big(\sqrt[2k-2]{\log_2 n}\big).$$

For $k = 2$ and $n \to \infty$ we a.a.s. have

$$\chi(KG_{n,k}(p)) = n - \Theta\big(\sqrt[2]{\log_2 n \cdot \log_2 \log_2 n}\big).$$

For $k = 1$, $KG_{n,k}$ is just the complete graph K_n, and thus $KG_{n,1}(p) = G(n, p)$. Therefore, we a.a.s. have $\chi(KG_{n,1}(p)) = \Theta(\frac{n}{\log n})$ (see, e.g., [**2**]), that is, an analogue of Theorem 1 cannot hold.

We note that a weaker version of Theorem 1 was announced in the short note due to the first author and Raigorodskii [**20**].

While the methods for studying Kneser graphs and hypergraphs in [**22**], [**1**] were topological, in [**23**] combinatorial methods relating the structure of $KG_{n,k}$ and $KG_{n,k+l}$ were used. In this paper we use (different) combinatorial and probabilistic methods, which are based on the analysis of the structure of independent sets in $KG_{n,k}(p)$ and some parts of which are somewhat resemblant of [**7**], [**3**]. In particular, we prove the following lemma about the structure of colourings of random Kneser graph.

Lemma 2. *For any fixed $0 < p < 1$, $k \geq 2$, $n \to \infty$ and some some $C = C(p, k)$ the random graph $KG_{n,k}(p)$ a.a.s. has a colouring in $\chi(KG_{n,k}(p))$ colours such that at least $n - C\sqrt[k-1]{\log_2 n}$ of colour classes form a subset of a star.*

The papers [**22**], [**1**], [**23**] were also concerned with the following question: when does the chromatic number drop by at most an additive constant? The best results here are due to the second author [**23**], who proved the following a.a.s. bound for any fixed $l \geq 2$ and some absolute constant $C = C(l)$:

$$\chi(KG_{n,k}(1/2)) \geq n - 2k + 2 - 2l \quad \text{if} \quad k \geq C(n \log n)^{1/l}. \tag{3}$$

In this paper, we provide a major improvement of (3), replacing the polynomial dependence of k on n by logarithmic.

Theorem 3. *For any $l \geq 6$ there exists $C = C(l)$, such that for $n \to \infty$ a.a.s.*

$$\chi(KG_{n,k}(1/2)) \geq n - 2k + 2 - 2l \quad \text{if} \quad k \geq C \log^{\frac{1}{2l-3}} n. \tag{4}$$

[1]asymptotically almost surely, i.e., with probability tending to 1 as $n \to \infty$.

We, however, expect that the same result should hold for $k \geq C \log\log n$. We can prove an upper bound of this form (see [**23**, Section 5]). The difficulty is in the lower bound, and it is related to certain extremal properties of complexes.

Colourings and simplicial complexes

One of the difficulties that arise in the study of colourings of random Kneser graphs is that we poorly understand that happens if we reduce the number of colours by some constant value.

If we colour $KG_{n,k}$ into $\chi(KG_{n,k}) - 1 = n - 2k + 1$ colours, then we of course get at least 1 monochromatic edge. But the intuition suggests that we should have much more. Modifying the standard colouring of $KG_{n,k}$ by colouring subsets on the last $2k$ elements (instead of $2k-1$) in the same colour, we get a colouring with $\frac{1}{2}\binom{2k}{k}$ monochromatic edges.

Problem 4. Given k and n, what is the minimum number ζ of monochromatic edges in the colouring of $KG_{n,k}$ into $n - 2k + 1$ colours?

One approach that allows us to get some bounds on ζ is via *Schrijver graphs* $SG_{n,k}$, that is, induced subgraphs of Kneser graphs on k-sets not containing two cyclically consecutive elements of $[n]$. It is known that $\chi(SG_{n,k}) = \chi(KG_{n,k})$. However, the number of vertices in $SG_{n,k}$ is roughly $\binom{n-k}{k}$. Thus, taking an $n - 2k + 1$-colouring of $KG_{n,k}$ and a permutation of $[n]$, we get monochromatic edges in each induced $SG_{n,k}$ that corresponds to that permutation. Averaging over the choice of the permutaion, we can conclude that there are at least $\frac{|E(KG_{n,k})|}{|E(SG_{n,k})|}$ monochromatic edges in $KG_{n,k}$. This gives good bounds for $n = ck$ for constant c, but already for $n = k^2$ the aforementioned ratio is a constant independent of k.

It may be even more natural to study the vertex version of this problem. We believe that the following strengthening of the Lovász' bound $\chi(KG_{n,k}) \geq n - 2k + 2$ should be true.

Conjecture 1. The largest subset of vertices of $KG_{n,k}$ that may be properly coloured in $n - 2k + 1$ colours has size at most $\binom{n}{k} - c^k$, where $c > 1$ is some constant.

Again, if one modifies the standard colouring of $KG_{n,k}$, by first taking $n - 2k$ stars and then an intersecting family on the remaining set $\binom{X}{k}$, $|X| = 2k$, then the number of not coloured sets is $\binom{2k-1}{k} \approx 4^k$. Interestingly, we can do better and provide an example with only 3^k "missing" sets.

This question has a very interesting topological counterpart. We advise the reader to consult the book of Matoušek [**26**] for the introduction to topological method. A simplicial complex $\mathcal{H} \subset 2^{[n]}$ is a family satisfying the condition that if $H \in \mathcal{H}$ and $H' \subset H$, then $H' \in \mathcal{H}$. Put

$$\mathcal{H}^{*2}_{\Delta} := \big\{(H_1 \times \{1\}) \cup (H_2 \times \{2\}) : H_1, H_2 \in \mathcal{H}, H_1 \cap H_2 = \emptyset\big\}.$$

There is a natural free $\mathbb{Z}_2$-action on $\mathcal{H}_\Delta^{*2}$, and we can define the $\mathbb{Z}_2$-index $\mathrm{ind}_{\mathbb{Z}_2}(\mathcal{H}_\Delta^{*2})$ of $\mathcal{H}_\Delta^{*2}$ as the minimal dimension d of a sphere S^d, for which there exists a continuous map $\|\mathcal{H}_\Delta^{*2}\| \to S^d$ that commutes with the $\mathbb{Z}_2$-actions on the spaces. Using a result due to Sarkaria (see [**26**, Theorem 5.8.2]), Conjecture 1 is implied by the following conjecture:

Conjecture 2. There exists $c > 1$, such that if $\mathcal{H}$ is a simplicial complex that has fewer than c^k k-element sets, then $\mathrm{ind}_{\mathbb{Z}_2}(\mathcal{H}_\Delta^{*2}) \le 2k-3$ (or even the following is true: $\mathcal{H}$ is embeddable into $\mathbb{R}^{2k-3}$).

Note that, substituting $c = 1$ in the (version in the brackets of the) conjecture above, we get the geometric realisation theorem, stating that any finite $k-1$-dimensional simplicial complex has a geometric realisation in $\mathbb{R}^{2k-3}$.

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