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NEW PROPERTIES OF PROLONGATIONS OF LINEAR CONNECTIONS ON WEIL BUNDLES

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Abstract. Let M be a paracompact smooth manifold, A be a Weil algebra and M^A be the associated Weil bundle. If ∇ is a linear connection on M, we give equivalent definition and the properties of the prolongation ∇^A to M^A equivalent to the prolongation defined by Morimoto. When (M, g) is a pseudo-riemannian manifold, we show that the symmetric tensor g^A of type $(0,2)$ defined by Okassa is nondegenerated. At the end, we show that, if ∇ is a Levi-Civita connection on (M, g), then ∇^A is torsion-free and g^A is parallel with respect to ∇^A.

1. Introduction

In what follows, we denote A a local algebra (in the sense of André Weil) or simply Weil algebra, M a smooth manifold, $C^\infty(M)$ the algebra of smooth functions on M and M^A the manifold of infinitely near points of kind A [**10**]. The triplet (M^A, π, M) is a bundle called bundle of infinitely near points or simply Weil bundle.

If $f\colon M \to \mathbb{R}$ is a smooth function, then the application

$$f^A\colon M^A \to A, \qquad \xi \mapsto \xi(f),$$

is also a smooth function. The set $C^\infty(M^A, A)$ of smooth functions on M^A with values in A is a commutative algebra over A with unit and the application

$$C^\infty(M) \to C^\infty(M^A, A), \qquad f \mapsto f^A,$$

is an injective homomorphism of algebras. Then, we have

$$(f+g)^A = f^A + g^A; \qquad (\lambda \cdot f)^A = \lambda \cdot f^A; \qquad (f \cdot g)^A = f^A \cdot g^A.$$

The map

$$C^\infty(M^A) \times A \to C^\infty(M^A, A), \qquad (F, a) \mapsto F \cdot a\colon \xi \mapsto F(\xi) \cdot a$$

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is bilinear and induces one and only one linear map

$$\sigma\colon C^\infty(M^A)\otimes A\to C^\infty(M^A,A).$$

When $(a_\alpha)_{\alpha=1,2,\dots,\dim A}$ is a basis of A and $(a^*_\alpha)_{\alpha=1,2,\dots,\dim A}$ is a dual basis of the basis $(a_\alpha)_{\alpha=1,2,\dots,\dim A}$, the application

$$\sigma^{-1}\colon C^\infty(M^A,A)\to A\otimes C^\infty(M^A),\qquad \varphi\mapsto\sum_{\alpha=1}^{\dim A}a_\alpha\otimes(a^*_\alpha\circ\varphi),$$

is an isomorphism of A-algebras. That isomorphism does not depend of a chosen basis and the application

$$\gamma\colon C^\infty(M)\to A\otimes C^\infty(M^A),\qquad f\mapsto\sigma^{-1}(f^A)$$

is a homomorphism of algebras.

If (U,φ) is a local chart of M with local coordinate system $(x_1,\dots,x_n)$, the map

$$\varphi^A\colon U^A\to A^n,\qquad \xi\mapsto(\xi(x_1),\dots,\xi(x_n))$$

is a bijection from U^A onto an open set of A^n. In addition, if $(U_i,\varphi_i)_{i\in I}$ is an atlas of M^A, then $(U_i^A,\varphi_i^A)_{i\in I}$ is also an A-atlas of M^A [**2**].

1.1. Vector fields on M^A

In [**6**], we presented another characterization of a vector field on M^A through the above theorem and also gave a writing of a vector field on M^A in coordinate neighborhood system.

Thus, we have the next two theorems.

Theorem 1. *The following assertions are equivalent:*

1. *A vector field on M^A is a differentiable section of the tangent bundle (TM^A,π_{M^A},M^A).*
2. *A vector field on M^A is a derivation of $C^\infty(M^A)$.*
3. *A vector field on M^A is a derivation of $C^\infty(M^A,A)$ which is A-linear.*
4. *A vector field on M^A is a linear map $X\colon C^\infty(M)\to C^\infty(M^A,A)$ such that*

$$X(f\cdot g)=X(f)\cdot g^A+f^A\cdot X(g)\qquad\text{for any } f,g\in C^\infty(M).$$

We verify that the $C^\infty(M^A,A)$-module $\mathfrak{X}(M^A)$ of vector fields on M^A is a Lie algebra over A.

Theorem 2. *The map*

$$\mathfrak{X}(M^A)\times\mathfrak{X}(M^A)\to\mathfrak{X}(M^A),\qquad (X,Y)\mapsto[X,Y]=X\circ Y-Y\circ X,$$

is skew-symmetric A-bilinear and defines a structure of A-Lie algebra over $\mathfrak{X}(M^A)$.

In the following, we look at a vector field as an A-linear map

$$X\colon C^\infty(M^A,A)\to C^\infty(M^A,A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi) \qquad \text{for any } \varphi, \psi \in C^\infty(M^A, A),$$

that is to say,

$$\mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)].$$

1.2. Prolongations to M^A of vector fields on M

Proposition 3. *If* $\theta\colon C^\infty(M) \to C^\infty(M)$ *is a vector field on* M*, then there exists one and only one* A*-linear derivation*

$$\theta^A\colon C^\infty(M^A, A) \to C^\infty(M^A, A)$$

such that $\theta^A(f^A) = [\theta(f)]^A$ *for any* $f \in C^\infty(M)$*. Thus, if* $\theta, \theta_1, \theta_2$ *are vector fields on* M *and* $f \in C^\infty(M)$*, then we have*

(1) $$(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A; \quad (f \cdot \theta)^A = f^A \cdot \theta^A \quad \textit{and} \quad [\theta_1, \theta_2]^A = \left[\theta_1^A, \theta_2^A\right].$$

2. Prolongation of linear connections on Weil bundles

In this section, if ∇ [**3**] is a linear connection on M, we give equivalent definition and the properties of the prolongation ∇^A to M^A equivalent to the prolongation $\overline{\nabla}$ defined by Morimoto [**5**]. When (M, g) is a pseudo-riemannian manifold, we show that the symmetric tensor g^A of type $(0,2)$ defined by Okassa is nondegenerated [**7**]. At the end, we show that if ∇ is a Levi-Civita connection on (M, g), then ∇^A is torsion-free and g^A is parallel with respect to ∇^A.

According to [**6**], if $X\colon M^A \to TM^A$ is a vector field on M^A and U is a coordinate neighborhood of M with the coordinate neighborhood $(x_1, \dots, x_n)$, then there exist some functions $f_i \in C^\infty(U^A, A)$ for $i = 1, \dots, n$ such that

$$X_{|U^A} = \sum_{i=1}^n f_i \left(\frac{\partial}{\partial x_i}\right)^A.$$

When (U, φ) is a local chart and $(x_1, \dots, x_n)$ his local coordinate system, then the map

$$U^A \to A^n, \qquad \xi \mapsto (\xi(x_1), \dots, \xi(x_n)),$$

is a diffeomorphism from U^A onto an open set of A^n. As

$$\left(\frac{\partial}{\partial x_i}\right)^A : C^\infty(U^A, A) \to C^\infty(U^A, A)$$

is such that $\left(\frac{\partial}{\partial x_i}\right)^A (x_j^A) = \delta_{ij}$, we can denote $\frac{\partial}{\partial x_i^A} = \left(\frac{\partial}{\partial x_i}\right)^A$. If $v \in T_\xi M^A$, we can write

$$v = \sum_{i=1}^n v(x_i^A)] \left.\frac{\partial}{\partial x_i^A}\right|_\xi.$$

If $X \in \mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)]$, we have

$$X|_{U^A} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i^A}$$

with $f_i \in C^\infty(U^A, A)$ for $i = 1, 2, \ldots, n$.

2.1. Equivalent definitions of derivation laws in $\mathfrak{X}(M^A)$.

In this subsection, we give the definitions of a derivation law in $\mathfrak{X}(M^A) = \mathrm{Der}_\mathbb{R}[C^\infty(M^A)]$ and a derivation law in $\mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)]$.
Let R be an algebra over a commutative field $\mathbb{K}$. We recall that a derivation law in a R-module P is a map

$$D\colon \mathrm{Der}_\mathbb{K}(R) \to \mathrm{End}_\mathbb{K}(P)$$

such that:

1. D is R-linear;
2. For any $d \in \mathrm{Der}_\mathbb{K}(R)$, the $\mathbb{K}$-endomorphism $D_d\colon P \to P$ satisfies

$$D_d(r \cdot p) = d(r) \cdot p + r \cdot D_d(p)$$

for any $r \in R$ and any $p \in P$, see [**4**].

We also recall that a derivation law in the $C^\infty(M)$-module $\mathfrak{X}(M) = \mathrm{Der}_\mathbb{R}[C^\infty(M)]$ of vector fields on M is a map

$$D\colon \mathfrak{X}(M) = \mathrm{Der}_\mathbb{R}[C^\infty(M)] \to \mathrm{End}_\mathbb{R}[\mathfrak{X}(M) = \mathrm{Der}_\mathbb{R}[C^\infty(M)]]$$

such that:

1. D is $C^\infty(M)$-linear;
2. For any $\theta \in \mathfrak{X}(M)$, the $\mathbb{R}$-endomorphism $D_\theta\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfies

$$D_\theta(f \cdot \mu) = \theta(f) \cdot \mu + f \cdot D_\theta(\mu)$$

for any $f \in C^\infty(M)$, and any $\mu \in \mathfrak{X}(M^A)$.

Derivation law defines a linear connection on M, see [**9**].

Definition 1. A derivation law in $\mathfrak{X}(M^A) = \mathrm{Der}_\mathbb{R}[C^\infty(M^A)]$ is a map

$$D\colon \mathfrak{X}(M^A) = \mathrm{Der}_\mathbb{R}[C^\infty(M^A)] \to \mathrm{End}_\mathbb{R}\left[\mathfrak{X}(M^A) = \mathrm{Der}_\mathbb{R}\left[C^\infty(M^A)\right]\right]$$

such that:

1. D is $C^\infty(M^A)$-linear;
2. For any $X \in \mathfrak{X}(M^A)$, the $\mathbb{R}$-endomorphism $D_X : \mathfrak{X}(M^A) \to \mathfrak{X}(M^A)$ satisfies

$$D_X(F \cdot Y) = X(F) \cdot Y + F \cdot D_X(Y)$$

for any $F \in C^\infty(M^A)$ and any $Y \in \mathfrak{X}(M^A)$.

Another definition.

In what follows, we denote $\mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)]$.
We denote $\mathrm{End}_A[\mathfrak{X}(M^A)]$ the set of A-endomorphisms of $\mathfrak{X}(M^A)$, i.e., the set of maps from $\mathfrak{X}(M^A)$ into $\mathfrak{X}(M^A)$ which are linear over A.

Proposition 4. *The set* $\mathrm{End}_A[\mathfrak{X}(M^A)]$ *is a* $C^\infty(M^A, A)$*-module.*

Definition 2. A derivation law in $\mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)]$ is a map

$$D\colon \mathfrak{X}(M^A) \to \mathrm{End}_A\left[\mathfrak{X}(M^A)\right]$$

such that

1. D is $C^\infty(M^A, A)$-linear;
2. For any $X \in \mathfrak{X}(M^A)$, the A-endomorphism $D_X : \mathfrak{X}(M^A) \to \mathfrak{X}(M^A)$ verifies
$$D_X(\varphi \cdot Y) = X(\varphi) \cdot Y + \varphi \cdot D_X(Y)$$
for any $\varphi \in C^\infty(M^A)$ and any $Y \in \mathfrak{X}(M^A)$.

2.2. The new statement of the Morimoto's prolongation of a linear connection on M

Theorem 5. *If* ∇ *is a linear connection on* M*, then there exists one and only one linear application*

$$\nabla^A\colon \mathfrak{X}(M^A) \to \mathrm{End}_A[\mathfrak{X}(M^A)], \qquad X \mapsto \nabla^A_X,$$

such that

$$\nabla^A_{\theta^A}\eta^A = (\nabla_\theta \eta)^A$$

for any $\theta, \eta \in \mathfrak{X}(M)$.

Proof. If $X \in \mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)$, then

$$X(f^A) = \sum_{\alpha=1}^{\dim A} X'(a^*_\alpha \circ f^A) \cdot a_\alpha = \sum_{\alpha=1}^{\dim A} X(a^*_\alpha \circ f^A) \cdot a_\alpha$$

with $X' \in \mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A)]$.
Let

$$\overline{\nabla}\colon \mathfrak{X}(M^A) = \mathrm{Der}_{\mathbb{R}}[C^\infty(M^A)] \to \mathrm{End}_{\mathbb{R}}\left[\mathfrak{X}(M^A) = \mathrm{Der}_{\mathbb{R}}[C^\infty(M^A)]\right]$$

be the Morimoto's prolongation to M^A of the linear connection ∇ on M. We denote

$$\nabla^A\colon \mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)] \to \mathrm{End}_A\left[\mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)]\right]$$

the same derivation law in $\mathfrak{X}(M^A) = \mathrm{Der}_A[C^\infty(M^A, A)]$. Thus, for any θ, $\eta \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\left[\nabla^A_{\theta^A}\eta^A\right](f^A) &= \sum_{\alpha=1}^{\dim A}\left[\nabla^A_{\theta^A}\eta^A\right]'(a^*_\alpha \circ f^A)\cdot a_\alpha = \sum_{\alpha=1}^{\dim A}\left[\nabla^A_{(\theta^A)'}(\eta^A)'\right](a^*_\alpha \circ f^A)\cdot a_\alpha \\
&= \sum_{\alpha=1}^{\dim A}\left[(\nabla_\theta \eta)^A\right]'(a^*_\alpha \circ f^A)\cdot a_\alpha \\
&= \sum_{\alpha=1}^{\dim A}\left[(\nabla_\theta \eta)^A\right](a^*_\alpha \circ f^A)\cdot a_\alpha \\
&= \left[(\nabla_\theta \eta)^A\right](f^A)
\end{aligned}$$

for any $f \in C^\infty(M)$, hence,

$$\nabla^A_{\theta^A}\eta^A = (\nabla_\theta \eta)^A.$$

□

3. Prolongation of tensors to M^A

We denote $T^{p,q}(M^A)$ the $C^\infty(M^A, A)$-module of tensors of type (p, q) on M^A, i.e., the $C^\infty(M^A, A)$-module of multilinear applications of

$$\left[\mathfrak{X}\left(M^A\right)\right]^q \to \otimes^p_{C^\infty(M^A,A)}\mathfrak{X}\left(M^A\right)$$

and

$$\begin{aligned}
W^{p,q}\left(M^A\right) = \Big\{&\beta \in \mathfrak{L}^q\left(\mathfrak{X}(M), \otimes^p_{C^\infty(M^A,A)}\mathfrak{X}\left(M^A\right)\right) \\
&\mid \beta(f_1\theta_1, \ldots, f_q\theta_q) = f_1^A \cdots f_q^A \beta(\theta_1, \ldots, \theta_q)\Big\}
\end{aligned}$$

is a $C^\infty(M^A, A)$-module. Let

$$\gamma\colon C^\infty(M) \to C^\infty(M^A, A), \qquad f \to f^A;\ \ \mathfrak{X}(M) \to \mathfrak{X}\left(M^A\right), \qquad \theta \to \theta^A.$$

Theorem 6. *The map*

$$\kappa\colon T^{p,q}(M^A) \to W^{p,q}\left(M^A\right), \qquad \alpha \to \alpha \circ \gamma^q\colon\ (\theta_1, \ldots, \theta_q) \to \alpha\left(\theta_1^A, \ldots, \theta_q^A\right),$$

is an isomorphism of $C^\infty(M^A, A)$-modules.

Proof. 1. For any α_1, $\alpha_2 \in T^{p,q}(M^A)$, we have

$$\begin{aligned}
\left[\kappa(\alpha_1 + \alpha_2)\right](\theta_1, \ldots, \theta_q) &= [\alpha_1 + \alpha_2]\left(\theta_1^A, \ldots, \theta_q^A\right) \\
&= \alpha_1\left(\theta_1^A, \ldots, \theta_q^A\right) + \alpha_2\left(\theta_1^A, \ldots, \theta_q^A\right) \\
&= [\kappa(\alpha_1)](\theta_1, \ldots, \theta_q) + [\kappa(\alpha_2)](\theta_1, \ldots, \theta_q) \\
&= [\kappa(\alpha_1) + \kappa(\alpha_2)](\theta_1, \ldots, \theta_q).
\end{aligned}$$

2. For any $\alpha \in T^{p,q}(M^A)$ and any $\varphi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}[\kappa(\varphi\alpha)](\theta_1, \ldots, \theta_q) &= [\varphi\alpha]\left(\theta_1^A, \ldots, \theta_q^A\right) \\ &= \varphi\alpha\left(\theta_1^A, \ldots, \theta_q^A\right) \\ &= \varphi[\kappa(\alpha)](\theta_1, \ldots, \theta_q).\end{aligned}$$

□

Lemma 7. *Any $\alpha \in W^{p,q}(M^A)$ has a restriction at any open set U of M, i.e., if*

$$\theta_1 |_U = \theta_2 |_U \ldots = \theta_q |_U,$$

then

$$\alpha(\theta_1, \ldots, \theta_q) |_{U^A} = 0.$$

Proof. Let $\xi \in U^A$. Let $x_0 = \pi_A(\xi) \in U$ be origin of ξ.

There exist a function $f \in C^\infty(M)$ and an open neighborhood V of x_0, $V \subset U$, such that $f = 0$ on V and $f = 1$ on $\complement U$.

For any $x \in U$, $(f\theta_i)(x) = f(x) \cdot \theta_i(x) = \theta_i(x)$ and for any $x \in \complement U$, $(f\theta_i)(x) = f(x) \cdot \theta_i(x) = \theta_i(x)$; then $f\theta_i = \theta_i$ for any $i = 1, 2, \ldots, q$.

We have $f^A = 0$ on V^A and $f^A = 1$ on $\complement U^A$. Thus

$$\begin{aligned}[\alpha(\theta_1, \ldots, \theta_q)](\xi) &= [\alpha(f_1\theta_1, \ldots, f_q\theta_q)](\xi) \\ &= f_1^A(\xi) \cdots f_q^A(\xi)[\alpha(\theta_1, \ldots, \theta_q)](\xi).\end{aligned}$$

As ξ is whichever in U^A, we get

$$\alpha(\theta_1, \ldots, \theta_q) |_{U^A} = 0.$$

□

For an open neighborhood U of M, we have

$$\alpha_U : \left[\mathfrak{X}\left(U^A\right)\right]^q \to \otimes^p_{C^\infty(U^A, A)} \mathfrak{X}\left(U^A\right)$$

with

$$\alpha_U(\theta_1 |_U, \ldots, \theta_q |_U) = \alpha(\theta_1, \ldots, \theta_q) |_{U^A}.$$

Proposition 8. *For a coordinate neighborhood $U \subset M$ with the coordinate system $(x_1, \ldots, x_n)$ and $\alpha \in W^{p,q}(M^A)$, then the map*

$$\beta_{U^A} : \left[\mathfrak{X}\left(U^A\right)\right]^q \to \otimes^p_{C^\infty(U^A, A)} \mathfrak{X}\left(U^A\right)$$

such that

$$\beta_{U^A}\left(\sum_{i_1} f^1_{i_1}\left(\frac{\partial}{\partial x_{i_1}}\right)^A, \ldots, \sum_{i_q} f^q_{i_q}\left(\frac{\partial}{\partial x_{i_q}}\right)^A\right) = \sum_{i_1, \ldots, i_q} f^1_{i_1} \cdots f^q_{i_q} \alpha_U\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_q}}\right)$$

is $C^\infty\left(U^A, A\right)$-multilinear. Therefore, β_{U^A} is a tensor of type (p, q) on U^A.

Proof. We verify there exists a tensor $\beta\colon \left[\mathfrak{X}\left(M^A\right)\right]^q \to \otimes^p_{C^\infty(M^A,A)}\mathfrak{X}\left(M^A\right)$ such that $\beta_{|U^A} = \beta_{U^A}$. Thus

$$\begin{aligned}
\beta\left(\theta_1^A,\dots,\theta_q^A\right)_{|U^A} &= \beta_{|U^A}\left(\theta_1^A\,|_{U^A},\dots,\theta_q^A\,|_{U^A}\right)\\
&= \beta_{U^A}\left(\theta_1^A\,|_{U^A},\dots,\theta_q^A\,|_{U^A}\right)\\
&= \beta_{U^A}\left(\sum_{i_1}\left(p^1_{i_1}\right)^A\left(\frac{\partial}{\partial x_{i_1}}\right)^A,\dots,\sum_{i_q}\left(p^q_{i_q}\right)^A\left(\frac{\partial}{\partial x_{i_q}}\right)^A\right)\\
&= \sum_{i_1,\dots,i_q}\left(p^1_{i_1}\right)^A\cdots\left(p^q_{i_q}\right)^A\beta_{U^A}\left(\left(\frac{\partial}{\partial x_{i_1}}\right)^A,\dots,\left(\frac{\partial}{\partial x_{i_q}}\right)^A\right)\\
&= \sum_{i_1,\dots,i_q}\left(p^1_{i_1}\right)^A\cdots\left(p^q_{i_q}\right)^A\alpha_U\left(\frac{\partial}{\partial x_{i_1}},\dots,\frac{\partial}{\partial x_{i_q}}\right)\\
&= \alpha_U\left(\sum_{i_1}p^1_{i_1}\frac{\partial}{\partial x_{i_1}},\dots,\sum_{i_q}p^q_{i_q}\frac{\partial}{\partial x_{i_q}}\right)\\
&= \alpha_U\left(\theta_1\,|_U,\dots,\theta_q\,|_U\right)\\
&= \alpha\left(\theta_1,\dots,\theta_q\right)_{|_{U^A}}.
\end{aligned}$$

As U is whichever, we deduce that

$$\beta\left(\theta_1^A,\dots,\theta_q^A\right) = \alpha\left(\theta_1,\dots,\theta_q\right).$$

□

Proof of the theorem. $[\sigma(\alpha)]\left(\theta_1,\dots,\theta_q\right) = \alpha\left(\theta_1^A,\dots,\theta_q^A\right)$, for $\alpha'\colon [\mathfrak{X}(M)]^q \to \otimes^p_{C^\infty(M^A,A)}\mathfrak{X}\left(M^A\right)$, we have

$$\beta\left(\theta_1^A,\dots,\theta_q^A\right) = \alpha\prime\left(\theta_1,\dots,\theta_q\right),$$

i.e., for any $\alpha\prime \in W^{p,q}(M^A)$, there exists one and only one $\alpha \in T^{p,q}(M^A)$ such that

$$\alpha\left(\theta_1^A,\dots,\theta_q^A\right) = \alpha\prime\left(\theta_1,\dots,\theta_q\right).$$

□

Theorem 9. *If $\alpha \in T^{p,q}(M)$ is a tensor of type (p,q) on M, then there exists one and only one tensor α^A of type (p,q) on M^A such that $\alpha^A \circ \gamma^q = \left(\overset{p}{\otimes}\gamma\right)\circ\alpha$.*

Corollary 10. If α is a tensor of type $(0,p)$ (of type $(1,p)$, respectively), then there exists one and only one tensor α^A of type $(0,p)$ (of type $(1,p)$, respectively), such that

$$\alpha\left(\theta_1^A,\dots,\theta_q^A\right) = \left[\alpha\left(\theta_1,\dots,\theta_q\right)\right]^A$$

for all $\theta_1,\dots,\theta_q$.

4. Prolongation of the Levi-Civita connection

In this section, we consider (M, g) a pseudo-riemannian manifold. In what follows we study the prolongation of connections to M^A deduced from the Levi-Civita connection on M.

Proposition 11 ([**7**])**.** *Let* $\mathrm{g}\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$ *be a symmetric tensor of type* $(0,2)$ *on* M. *There exists one and only one symmetric tensor* g^A *of type* $(0,2)$ *on* M^A *with value in* A *such that* $\mathrm{g}^A\left(a\cdot\eta^A, b\cdot\theta^A\right) = ab\cdot\left[\mathrm{g}\left(\eta,\theta\right)\right]^A$ *for any* $a, b\in A$ *and* $\eta,\theta\in\mathfrak{X}(M)$.

Following [**1**], we state

Proposition 12. *If* (M,g) *is a pseudo-riemannian manifold, then there exists one and only one* $C^\infty(M^A, A)$*-nondegenerated symmetric bilinear form*

$$\mathrm{g}^A\colon \mathfrak{X}(M^A)\times\mathfrak{X}(M^A)\to C^\infty(M^A, A)$$

such that for any vector fields η *and* θ *on* M,

$$\mathrm{g}^A\left(\eta^A,\theta^A\right) = \left[\mathrm{g}\left(\eta,\theta\right)\right]^A,$$

where η^A *and* θ^A*mean prolongations to* M^A *of vector fields* η *and* θ.

Proof. It is a matter here to show only the nondegeneracy of g^A, the proof is done in the same way as in [**1**]. □

Therefore, g^A is a pseudo-riemannian manifold on M^A and confers to M^A, the structure of a pseudo-riemannian manifold.

Let M be a smooth manifold endowed with a linear connection ∇. We denote T_∇ and R_∇ the torsion and the curvature, of ∇, respectively. If ∇^A is the prolongation of ∇ to M^A, T_{∇^A} and R_{∇^A} denote the torsion and the curvature of ∇^A, respectively.

Proposition 13. *We have* $T_{\nabla^A} = \left(T_\nabla\right)^A$.

Proof. $\left(T_\nabla\right)^A : \mathfrak{X}\left(M^A\right)\times\mathfrak{X}\left(M^A\right)\to\mathfrak{X}\left(M^A\right)$ is the unique tensor of type $(1,2)$ such that $\left(T_\nabla\right)^A\left(\theta_1^A,\theta_2^A\right) = \left[T_\nabla\left(\theta_1,\theta_2\right)\right]^A$. We have

$$T_{\nabla^A}\left(X,Y\right) = \nabla^A_X Y - \nabla^A_Y X - \left[X,Y\right].$$

Then

$$\begin{aligned} T_{\nabla^A}\left(\theta_1^A,\theta_2^A\right) &= \nabla^A_{\theta_1^A}\theta_2^A - \nabla^A_{\theta_2^A}\theta_1^A - \left[\theta_1^A,\theta_2^A\right] \\ &= \left(\nabla_{\theta_1}\theta_2\right)^A - \left(\nabla_{\theta_2}\theta_1\right)^A - \left[\theta_1,\theta_2\right]^A \\ &= \left(\nabla_{\theta_1}\theta_2 - \nabla_{\theta_2}\theta_1 - \left[\theta_1,\theta_2\right]\right)^A = \left[T_\nabla\left(\theta_1,\theta_2\right)\right]^A, \end{aligned}$$

hence,

$$T_{\nabla^A} = \left(T_\nabla\right)^A.$$

□

Proposition 14. *We have* $R_{\nabla^A} = (R_\nabla)^A$.

Proof. $(R_\nabla)^A : \mathfrak{X}\left(M^A\right) \times \mathfrak{X}\left(M^A\right) \times \mathfrak{X}\left(M^A\right) \to \mathfrak{X}\left(M^A\right)$ is the unique tensor of type $(1,3)$ such that $(R_\nabla)^A\left(\theta_1^A, \theta_2^A, \theta_3^A\right) = \left[R_\nabla\left(\theta_1, \theta_2, \theta_3\right)\right]^A$. We have:

$$R_{\nabla^A}(X,Y,Z) = \nabla_X^A\left(\nabla_Y^A Z\right) - \nabla_Y^A\left(\nabla_X^A Z\right) - \nabla_{[X,Y]}^A Z$$

Then

$$\begin{aligned} R_{\nabla^A}\left(\theta_1^A, \theta_2^A, \theta_3^A\right) &= \nabla_{\theta_1^A}^A\left(\nabla_{\theta_2^A}^A \theta_3^A\right) - \nabla_{\theta_2^A}^A\left(\nabla_{\theta_1^A}^A \theta_3^A\right) - \nabla_{\left[\theta_1^A, \theta_2^A\right]}^A \theta_3^A \\ &= \nabla_{\theta_1^A}^A\left(\nabla_{\theta_2}\theta_3\right)^A - \nabla_{\theta_2^A}^A\left(\nabla_{\theta_1}\theta_3\right)^A - \nabla_{[\theta_1,\theta_2]^A}^A \theta_3^A \\ &= \left[\nabla_{\theta_1}\left(\nabla_{\theta_2}\theta_3\right)\right]^A - \left[\nabla_{\theta_2}\left(\nabla_{\theta_1}\theta_3\right)\right]^A - \left[\nabla_{[\theta_1,\theta_2]}\theta_3\right]^A \\ &= \left[R_\nabla\left(\theta_1, \theta_2, \theta_3\right)\right]^A, \end{aligned}$$

hence,

$$R_{\nabla^A} = (R_\nabla)^A.$$

□

Proposition 15. *If* $h\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ *is a tensor of type* $(0,2)$*, then* $\nabla_{\theta^A}^A h^A = (\nabla_\theta h)^A$.

Proof. $h^A\colon \mathfrak{X}\left(M^A\right) \times \mathfrak{X}\left(M^A\right) \to C^\infty\left(M^A, A\right)$ is a tensor of type $(0,2)$ such that $h^A\left(\theta_1^A, \theta_2^A\right) = \left[h\left(\theta_1, \theta_2\right)\right]^A$. As

$$\nabla_\theta h\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

is a tensor of type $(0,2)$ on M, then there exists

$$(\nabla_\theta h)^A : \mathfrak{X}\left(M^A\right) \times \mathfrak{X}\left(M^A\right) \to C^\infty\left(M^A, A\right)$$

such that

$$(\nabla_\theta h)^A\left(\theta_1^A, \theta_2^A\right) = \left[(\nabla_\theta h)\left(\theta_1, \theta_2\right)\right]^A.$$

Indeed, we have

$$\left(\nabla_{\theta^A}^A h^A\right)(X,Y) = \theta^A\left[h^A(X,Y)\right] - h^A\left(\nabla_{\theta^A}^A X, Y\right) - h^A\left(X, \nabla_{\theta^A}^A Y\right)$$

for any $X, Y \in \mathfrak{X}\left(M^A\right)$, then for $\theta_1, \theta_2 \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \left(\nabla_{\theta^A}^A h^A\right)\left(\theta_1^A, \theta_2^A\right) &= \theta^A\left[h^A\left(\theta_1^A, \theta_2^A\right)\right] - h^A\left(\nabla_{\theta^A}^A \theta_1^A, \theta_2^A\right) - h^A\left(\theta_1^A, \nabla_{\theta^A}^A \theta_2^A\right) \\ &= \left[\theta\left[h\left(\theta_1, \theta_2\right)\right] - h\left(\nabla_\theta \theta_1, \theta_2\right) - h\left(\theta_1, \nabla_\theta \theta_2\right)\right]^A \\ &= \left[(\nabla_\theta h)\left(\theta_1, \theta_2\right)\right]^A, \end{aligned}$$

hence,

$$\nabla_{\theta^A}^A h^A = (\nabla_\theta h)^A.$$

□

Lemma 16. *The map* $\tau_g\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$, $(\theta,\theta_1,\theta_2)\mapsto (\nabla_\theta g)(\theta_1,\theta_2)$ *is a tensor of type* $(0,3)$. *Moreover, the map* $\tau_{g^A}\colon \mathfrak{X}\left(M^A\right)\times\mathfrak{X}\left(M^A\right)\times\mathfrak{X}\left(M^A\right)\to C^\infty\left(M^A,A\right)$, $(X,Y_1,Y_2)\mapsto \left(\nabla^A_X g^A\right)(Y_1,Y_2)$ *is also a tensor of type* $(0,3)$ *and* $\tau_{g^A}=(\tau_g)^A$.

Proof. For any $\theta,\theta_1,\theta_2\in\mathfrak{X}(M)$ and for $f_1,f_2,f_3\in C^\infty(M)$, we have

$$\begin{aligned}\tau_g(f_1\theta,f_2\theta_1,f_3\theta_2) &= \left(\nabla_{(f_1\theta)}g\right)(f_2\theta_1,f_3\theta_2)\\ &= f_1\theta\left[g(f_2\theta_1,h\theta_2)\right]-g(\nabla_{f_1\theta}f_2\theta_1,h\theta_2)-g(f_2\theta_1,\nabla_{f_1\theta}h\theta_2)\\ &= f_1\cdot\theta(f_2f_3)\cdot g(\theta_1,\theta_2)+f_1f_2f_3\cdot g(\theta_1,\theta_2)-g\big((f_1\theta)(f_2)\cdot\theta_1\\ &\quad +f_1\nabla_\theta\theta_1,h\theta_2\big)-1g\left(f_2\theta_1,(f_1\theta)(f_3)\cdot\theta_2+f_3f_1\nabla_\theta\theta_2\right)\\ &= f_1\cdot\theta(f_2f_3)\cdot g(\theta_1,\theta_2)+f_1f_2f_3\cdot g(\theta_1,\theta_2)-f_1\theta(f_2)\cdot g(\theta_1,\theta_2)\\ &\quad -f_2f_3f_1g(\nabla_\theta\theta_1,\theta_2)-f_2f_1\theta(h)\cdot g(\theta_1,\theta_2)-f_2f_3f_1g(\theta_1,\nabla_\theta\theta_2)\\ &= f_1f_2f_3\tau_g(\theta,\theta_1,\theta_2)\,.\end{aligned}$$

As τ_g is a tensor of type $(0,3)$, then there exists a tensor of type $(0,3)$

$$(\tau_g)^A:\mathfrak{X}\left(M^A\right)\times\mathfrak{X}\left(M^A\right)\times\mathfrak{X}\left(M^A\right)\to C^\infty\left(M^A,A\right)$$

such that $(\tau_g)^A\left(\theta^A,\theta_1^A,\theta_2^A\right)=\left[(\nabla_\theta g)(\theta_1,\theta_2)\right]^A$. Thus, we have

$$\begin{aligned}(\tau_g)^A\left(\theta^A,\theta_1^A,\theta_2^A\right) &= \left(\nabla^A_{\theta^A}g^A\right)\left(\theta_1^A,\theta_2^A\right)\\ &= \left[(\nabla_\theta g)(\theta_1,\theta_2)\right]^A\end{aligned}$$

what implies

$$\tau_{g^A}=(\tau_g)^A\,.$$

□

Proposition 17. *If* (M,g) *is a pseudo-riemannian manifold with canonical connection* ∇, *then* $\left(M^A,g^A\right)$ *is a pseudo-riemannian manfold with canonical connection* ∇^A, *i.e.,* $T_{\nabla^A}=0$ *and* $\nabla^A_X g^A=0$ *for any* $X\in\mathfrak{X}\left(M^A\right)$.

Proof. Let ∇^A be the prolongation of ∇. As $T_{\nabla^A}=(T_\nabla)^A$ and since $T_\nabla=0$, then $T_{\nabla^A}=0$. We also have $\tau_{g^A}(X,Y_1,Y_2)=\left(\nabla^A_X g^A\right)(Y_1,Y_2)$ for any $X,Y_1,Y_2\in\mathfrak{X}\left(M^A\right)$. As $\nabla_\theta g=0$, then $(\tau_g)^A=0=\tau_{g^A}$ what implies $\left(\nabla^A_X g^A\right)(Y_1,Y_2)=0$ for any $X,Y_1,Y_2\in\mathfrak{X}\left(M^A\right)$, hence, $\nabla^A_X g^A=0$ for any $X\in\mathfrak{X}\left(M^A\right)$. □

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