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ASYMMETRIC RAMSEY PROPERTIES OF RANDOM GRAPHS INVOLVING CLIQUES AND CYCLES

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Abstract. We prove that for every $\ell, r \geq 3$, there exists $c > 0$ such that for $p \leq cn^{-1/m_2(K_r, C_\ell)}$, with high probability there is a 2-edge-colouring of the random graph $\mathbf{G}_{n,p}$ with no monochromatic copy of K_r of the first colour and no monochromatic copy of C_ℓ of the second colour. This is a progress on a conjecture of Kohayakawa and Kreuter.

1. Introduction

We say that a graph G is a *Ramsey graph for the pair of graphs* (F, H) if, in every 2-edge-colouring of G, we can find either a copy of F in which all the edges have the first colour or a copy of H in which all the edges have the second colour. In this case, we write $G \to (F, H)$. When $F = H$, we simplify the notation by just writing $G \to F$. Ramsey's Theorem [**7**] implies that, for every pair of graphs (F, H), there exists a graph G such that $G \to (F, H)$.

A lot of research has been devoted to understand the structure of Ramsey graphs. For example, Erdős and Hajnal [**1**] asked to determine positive integers k for which there exists G containing no copy of K_{k+1} and such that $G \to K_k$. Folkman [**2**] proved that such G exists for all k. Nešetřil and Rödl [**6**] proved a more general result which states that, for every F, there exists G with the same clique number as F such that $G \to F$. Rödl and Ruciński [**8**] proved that the binomial random graph $\mathbf{G}_{n,p}$ with high probability (w.h.p.) is a Ramsey graph for F, for certain range of $p = p(F)$. More precisely, they showed the following.

Theorem 1 (Rödl, Ruciński, 1995)**.** *Let F be a graph containing a cycle. Then there exist positive constants c and C such that, for $p = p(n)$, we have*

$$\lim_{n\to\infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to F\big] = \begin{cases} 0, & \text{if } p \leq cn^{-1/m_2(F)}; \\ 1, & \text{if } p \geq Cn^{-1/m_2(F)}, \end{cases}$$

where

$$m_2(F) = \max\left\{ \frac{e(F') - 1}{v(F') - 2} : F' \subseteq F, v(F') \geq 3 \right\}.$$

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Therefore it is well understood when the random graph is a Ramsey graph for a fixed graph F. A natural generalisation of such a problem is to analyse for what values of $p = p(F,H)$ the random graph $\mathbf{G}_{n,p}$ is likely to be a Ramsey graph for a fixed pair of graphs (F,H). In this direction, Kohayakawa and Kreuter [**3**] conjectured the following.

Conjecture 2 (Kohayakawa, Kreuter, 1997)**.** Let F and H be graphs with $m_2(F) \geq m_2(H) > 1$. Then there exist positive constants c and C such that, for $p = p(n)$, we have

$$\lim_{n\to\infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to (F,H)\big] = \begin{cases} 0, & \text{if } p \leq cn^{-1/m_2(F,H)}; \\ 1, & \text{if } p \geq Cn^{-1/m_2(F,H)}, \end{cases}$$

where

$$m_2(F,H) = \max\left\{ \frac{e(F')}{v(F') - 2 + 1/m_2(H)} : F' \subseteq F, e(F) \geq 1 \right\}$$

Kohayakawa and Kreuter [**3**] proved that the conjecture holds in the case where F and H are both cycles and Marciniszyn, Skokan, Spöhel and Steger [**4**] proved that it holds when F and H are both complete graphs.

Here we establish the validity of Conjecture 2 when F is a clique and H is a cycle by proving the following theorem.

Theorem 3. *For all $\ell, r \geq 3$, there exists $c > 0$ such that for $p = p(n) \leq cn^{-1/m_2(K_r,C_\ell)}$, we have*

$$\lim_{n\to\infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to (K_r, C_\ell)\big] = 0.$$

We then combine Theorem 3 with the result from Mousset, Nenadov and Samotij [**5**], who proved that, for any pair of graphs (F,H) as in the Conjecture 2, $\lim_{n\to\infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to (F,H)\big] = 1$ for $p \geq Cn^{-1/m_2(F,H)}$.

2. Proof overview

In this section we shall give an overview of the proof of Theorem 3. Notice that we need to only consider case when $\ell, r \geq 4$; the remaining cases follow from [**3**] and [**4**].

Our proof strategy is similar to [**3**] and [**4**]. We first show that if $\mathbf{G}_{n,p} \to (K_r, C_\ell)$, for some $p \leq cn^{-1/m_2(K_r,C_\ell)}$ then w.h.p. we are able to execute a procedure on $\mathbf{G}_{n,p}$ which, w.h.p., will find some subgraph of $\mathbf{G}_{n,p}$ which is either very dense or it is very large and has a tree-like structure. We then show that $\mathbf{G}_{n,p}$, for that range of p, w.h.p., does not contain such subgraphs. While the overall strategy is similar to [**3**] and [**4**], the analysis of the procedure in the first step heavily depends on the pair (K_r, C_ℓ). In this point, our work differs from previous work. In order to describe the procedure, we introduce some notation in the following.

Given a graph $G = (V,E)$, we denote by $\mathcal{G}(G)$ the hypergraph whose hyperedges correspond to copies of K_r and C_ℓ on G. More precisely, $V\big(\mathcal{G}(G)\big) = E(G)$ and

$E(\mathcal{G}(G)) = \mathcal{E}_1 \cup \mathcal{E}_2$, where

$$\mathcal{E}_1 = \{E(F) : F \cong K_r, F \subseteq G\}$$
$$\mathcal{E}_2 = \{E(F) : F \cong C_\ell, F \subseteq G\}$$

Moreover, if $\mathcal{H}$ is a subhypergraph of $\mathcal{G}(G)$, we denote by $G(\mathcal{H})$ the underlying graph of G with edge set spanned by $\cup_{E\in E(\mathcal{H})}E$ and vertex set equal to $V(G)$. We also denote by $\mathcal{E}_i(\mathcal{H})$ the set of hyperedges of $\mathcal{H}$ belonging to $\mathcal{E}_i$. Then we have that $G \to (K_r, C_\ell)$ if, and only if, for every 2-colouring of the vertices of $\mathcal{G}(G)$, there exist a hyperedge $E \in \mathcal{E}_i(\mathcal{G})$, for some $i \in [2]$, such that every vertex in E has the colour i. We say that a hypergraph $\mathcal{H} \subseteq \mathcal{G}(G)$ is $\star$-critical if for any hyperedge $E \in \mathcal{E}_i(\mathcal{H})$, $i \in [2]$, and any hypervertex $e \in E$ there exists a hyperedge $F \in \mathcal{E}_{3-i}(\mathcal{H})$ such that $E \cap F = \{e\}$. The following simple (though maybe not immediately obvious) lemma connects Ramsey graphs to $\star$-critical hypergraphs.

Lemma 4. *If $G \to (K_r, C_\ell)$, then there exist $\mathcal{H} \subseteq \mathcal{G}(G)$ which is $\star$-critical.*

For a simple graph H, let $\lambda(H) = v(H) - \frac{e(H)}{m_2(K_r,C_\ell)}$. Notice that the expected number of copies of H in $\mathbf{G}_{n,p}$, for $p \le cn^{-1/m_2(K_r,C_\ell)}$, is at most $c^{e(H)} \cdot n^{\lambda(H)}$. In some sense, $\lambda(H)$ may be compared to the density of H. The following lemma, roughly speaking, states that $\star$-critical hypergraphs generated by $\mathbf{G}_{n,p}$ that do not have too may hyperedges must generate dense subgraphs in $\mathbf{G}_{n,p}$.

Lemma 5. *For all $\ell, r \ge 4$, there exist $\varepsilon_0, c > 0$ such that for $p = p(n) \le cn^{-1/m_2(K_r,C_\ell)}$, the following holds w.h.p. If $\mathcal{H} \subseteq \mathcal{G}(\mathbf{G}_{n,p})$ is $\star$-critical and has at most $\ell r^2 \log n$ hypervertices, then $\lambda(G(\mathcal{H})) \le -\varepsilon_0$.*

Algorithm 1, when applied to a $\star$-critical subhypergraph $\mathcal{G}_0 \subseteq \mathcal{G}(\mathbf{G}_{n,p})$, will create w.h.p. a sequence of subhypergraphs $\mathcal{H}_0 \subseteq \cdots \subseteq \mathcal{H}_i \subseteq \mathcal{G}_0$, each with a structure very close to a linear hypertree. The algorithm stops when the current hypergraph $\mathcal{H}_i$ is already too large or when the underlying graph $G(\mathcal{H}_i)$ is too dense. The first condition is quantified by the number of steps of the algorithm and the last condition is quantified by $\lambda(G(\mathcal{H}_i))$.

So in a step $i \le \log n$ with $\lambda(G(\mathcal{H}_i)) > -\varepsilon_0$, the Algorithm 1 will generate a hypergraph $\mathcal{H}' \nsubseteq \mathcal{H}_i$ with $v(\mathcal{H}') \le \ell r^2$ and let $\mathcal{H}_{i+1} = \mathcal{H}_i \cup \mathcal{H}'$. Depending on how $\mathcal{H}'$ was generated, we may have to consider this step as degenerated and in this case we add $i+1$ to the set DEG, which is an auxiliary set with the only purpose of tracking the degenerated steps. The way that we generate $\mathcal{H}'$ will depend on weather there is a hyperedge $E \in \mathcal{E}_1(\mathcal{G}_0)$ which intersects $G(\mathcal{H}_i)$ in at least two vertices and is not contained in $G(\mathcal{H}_i)$. This case distinction is done in line 4 of Algorithm 1. If such a hyperedge E exists, then $\mathcal{H}'$ will be simply $\{E\}$ and we consider this step degenerated. Otherwise, if we do not have such a hyperedge, then the procedure to generate $\mathcal{H}'$ is more intricate and we will not be able to describe it in detail here. But the idea is roughly the following. Since we have $e(\mathcal{H}_i) \le \ell r^2 \log n$ and $\lambda(G(\mathcal{H}_i)) > -\varepsilon_0$, Lemma 5 implies that w.h.p. $\mathcal{H}_i$ is not $\star$-critical. Then, because we failed the condition on line 4 of the Algorithm 1 together with the fact that $\mathcal{G}_0$ is $\star$-critical, we will be able to show that there exist a hyperedge $F \in \mathcal{E}_2(\mathcal{G}_0)$ which intersects $G(\mathcal{H}_i)$ in an edge and is not contained in $G(\mathcal{H}_i)$. Then $\mathcal{H}'$ will be built as an extension of F. Finally, if $\mathcal{H}$ adds too many vertices to $\mathcal{H}_i$, then we consider this step degenerated.

Algorithm 1

Input: a $\star$-critical subhypergraph $\mathcal{G}_0 \subseteq \mathcal{G} = \mathcal{G}(\mathbf{G}_{n,p})$
Output: a triple $(i, \mathcal{H}_i, \mathrm{DEG})$ where $\mathcal{H}_i \subseteq \mathcal{G}_0$ and $\mathrm{DEG} \subseteq [i]$

$i \leftarrow 0$
DEG $\leftarrow \emptyset$
Let $\mathcal{H}_0 = \{E_0\}$, where E_0 is any hyperdge from $\mathcal{E}_1(\mathcal{G}_0)$
while $i \le \log(n)$ and $\lambda(G(\mathcal{H}_i)) > -\varepsilon_0$ **do**
  if there exists $E \in \mathcal{E}_1(\mathcal{G}_0)$ such that $E \not\subseteq G(\mathcal{H}_i)$ and $|V(E) \cap V(G(\mathcal{H}_i))| \ge 2$ **then**
    $\mathcal{H}' \leftarrow \{E\}$
    DEG $\leftarrow$ DEG $\cup \{i+1\}$
  else
    $\langle\langle$ Compute $\mathcal{H}'$ $\rangle\rangle$
    if $\mathcal{H}'$ is degenerated **then**
      DEG $\leftarrow$ DEG $\cup \{i+1\}$
    end if
  end if
  $\mathcal{H}_{i+1} \leftarrow \mathcal{H}_i \cup \mathcal{H}'$
  $i \leftarrow i+1$
end while

In the following, we state claims that are sufficient to prove Theorem 3. We do not prove these claims here. While the proof of Claim 6 really depends on the fact that we are dealing with the pair (K_r, C_ℓ), the proofs of Claims 7 and 8 are general and follow the same argument of the corresponding lemmas in [**3**].

Claim 6. *For every $r, \ell \ge 4$, there exists $\delta > 0$ such that the following holds.*
(i) *If $i \in \mathrm{DEG}$, then $\lambda(G(\mathcal{H}_i)) \le \lambda(G(\mathcal{H}_{i-1})) - \delta$.*
(ii) *If $i \notin \mathrm{DEG}$, then $\lambda(G(\mathcal{H}_i)) = \lambda(G(\mathcal{H}_{i-1}))$.*
In particular, $\lambda(G(\mathcal{H}_i)) \le \lambda(K_r)$.

The following claim is actually a consequence of the previous claim.

Claim 7. *For every $r, \ell \ge 4$, there exists $M > 0$ such that for every output $(i, \mathcal{H}_i, \mathrm{DEG})$ of Algorithm 1, we have $|\mathrm{DEG}| \le M$.*

For all positive integers d and k, let $\mathcal{F}(d,k)$ be the family of all non-isomorphic graphs H such that $H = G(\mathcal{H}_i)$, where $\mathcal{H}_i$ comes from some possible output $(i, \mathcal{H}_i, \mathrm{DEG})$ of Algorithm 1 with $i \le k$ and $|\mathrm{DEG}| \le d$.

Claim 8. *For every $r, \ell \ge 4$, there exists $\alpha > 0$ such that for any $d, k \ge 1$, we have $|\mathcal{F}(d,k)| \le k^{\alpha d}$.*

Proof of Theorem 3. From Claim 7, we have that after applying Algorithm 1 to some $\star$-critical subhypergraph $\mathcal{G}_0 \subseteq \mathcal{G}(\mathbf{G}_{n,p})$, we get, w.h.p., as an output $(i, \mathcal{H}_i, \mathrm{DEG})$ with $i \leq \log n$ and $|\mathrm{DEG}| \leq M$. In particular, $H \subseteq \mathbf{G}_{n,p}$, for some $H \in \mathcal{F}(M, \log n)$. Therefore

$$\begin{aligned}\mathbb{P}\big[\mathbf{G}_{n,p} \to (K_r, C_\ell)\big] &\leq \mathbb{P}\big[\exists H \subseteq \mathbf{G}_{n,p} : H \in \mathcal{F}(M, \log n)\big] + o(1) \\ &\leq \sum_{H \in \mathcal{F}(M,\log n)} \mathbb{P}\big[H \subseteq \mathbf{G}_{n,p}\big] + o(1)\end{aligned}$$

The additional $o(1)$ term comes from the fact that Algorithm 1 will only generate an output with high probability.

Now for any $H \in \mathcal{F}(M, \log n)$, because of the condition in line 6 of the Algorithm 1, we have that either (i) $e(H) \geq \log n$ or (ii) $\lambda(H) \leq -\varepsilon_0$. In case (i), since $\lambda(H) \leq \lambda(K_r)$, we have

$$\mathbb{P}\big[H \subseteq \mathbf{G}_{n,p}\big] \leq c^{e(H)} n^{\lambda(H)} \leq c^{\log n} n^{\lambda(K_r)} \leq n^{-\varepsilon_0},$$

by choosing $c = c(\ell, r)$ small enough. In case (ii), we get

$$\mathbb{P}\big[H \subseteq \mathbf{G}_{n,p}\big] \leq c^{e(H)} n^{\lambda(H)} \leq n^{\lambda(H)} \leq n^{-\varepsilon_0}.$$

Therefore, by Claim 8, we get

$$\begin{aligned}\mathbb{P}\big[\mathbf{G}_{n,p} \to (K_r, C_\ell)\big] &\leq |\mathcal{F}(M, \log n)| \cdot n^{-\varepsilon_0} \\ &\leq (\log n)^{\alpha M} \cdot n^{-\varepsilon_0} = o(1).\end{aligned}$$

□

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