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FLOW NUMBER AND CIRCULAR FLOW NUMBER OF SIGNED CUBIC GRAPHS

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ABSTRACT. Let $\Phi(G,\sigma)$ and $\Phi_c(G,\sigma)$ denote the flow number and the circular flow number of a flow-admissible signed graph (G,σ), respectively. It is known that $\Phi(G) = \lceil\Phi_c(G)\rceil$ for every unsigned graph G. Based on this fact Raspaud and Zhu in 2011 conjectured that $\Phi(G,\sigma) - \Phi_c(G,\sigma) < 1$ holds also for every flow-admissible signed graph (G,σ). This conjecture was disproved by Schubert and Steffen using graphs with bridges and vertices of large degree. In this paper we focus on cubic graphs, since they play a crucial role in many open problems in graph theory. For cubic graphs we show that $\Phi(G,\sigma) = 3$ if and only if $\Phi_c(G,\sigma) = 3$ and if $\Phi(G,\sigma) \in \{4,5\}$, then $4 \le \Phi_c(G,\sigma) \le \Phi(G,\sigma)$. We also prove that all pairs of flow number and circular flow number that fulfill these conditions can be achieved in the family of bridgeless cubic graphs and thereby disprove the conjecture of Raspaud and Zhu even for bridgeless cubic signed graphs. Finally, we prove that all currently known graphs without nowhere-zero 5-flow have flow number and circular flow number 6 and propose several conjectures in this area.

1. Introduction

A signed graph (G,σ) is a graph G together with a signature σ assigning each edge a positive or a negative sign. Similarly to the unsigned graphs, flows on signed graphs follow a certain orientation of edges. Each edge consists of two half-edges, which are oriented independently, subject to the condition that a positive edge has its half-edges oriented consistently, while a negative edge has its half-edges oriented in the opposite direction. If a negative edge has its half-edges oriented from the incident vertices, it is called *introverted*, otherwise it is called *extroverted*. It can be easily seen that flows on unsigned graphs are equivalent to flows on signed graphs with all edges positive.

A *circular r-flow* (O,f) on (G,σ) is a pair where O is an orientation of $E(G)$, f is a function $E(G) \to \mathbb{R}$ such that $1 \le |f(e)| \le r-1$ for each edge $e \in E(G)$ and for every vertex $v \in V(G)$ the sum of incomnig flow-values equals the sum of outgoing flow-values.

A *nowhere-zero k-flow* is a circular k-flow such that every flow-value is an integer. The *flow number* $\Phi(G,\sigma)$ and the *circular flow number* $\Phi_c(G,\sigma)$ of a signed

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graph (G,σ) are the smallest integer k, and the smallest real number r, such that (G,σ) admits a nowhere-zero k-flow and a circular r-flow, respectively.

The research in the area of nowhere-zero flows on signed graphs has been stimulated by Bouchet's 6-flow conjecture [**1**], which states that if a signed graph admits a nowhere-zero flow, then it admits a 6-flow. It is known that the conjecture is true if the constant 6 is replaced with 12, see [**2**]. On the other hand, there are examples of signed graphs with flow number 6, see Fig. 1 and [**1, 5, 9**].

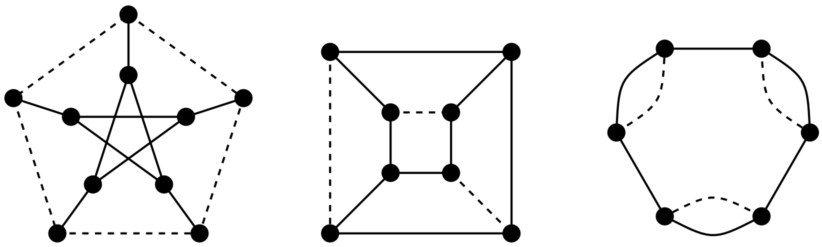

Figure 1. Signed graphs with flow number 6.

Clearly, $\Phi_c(G,\sigma) \leq \Phi(G,\sigma)$. In the unsigned case, the relationship between these two numbers is simple: $\Phi(G) = \lceil \Phi_c(G) \rceil$. This implies that $\Phi(G) - \Phi_c(G) < 1$. Raspaud and Zhu [**8**] conjectured that the same is true for all flow-admissible signed graphs and proved that

$$\Phi(G,\sigma) \leq 2\lceil \Phi_c(G,\sigma) \rceil - 1. \tag{*}$$

Schubert and Steffen [**9**] disproved the conjecture of Raspaud and Zhu and later Máčajová and Steffen [**7**] proved that the inequality (*) cannot be improved. Nevertheless, the results of [**9**] and [**7**] were obtained for signed graphs with bridges and vertices of large degree, leaving the conjecture of Raspaud and Zhu open for cubic graphs – an important family of graphs to which many flow related conjectures and problems can be reduced. In this paper, we partially fill this gap by proving the following theorem.

Theorem 1. *Let (G,σ) be a flow-admissible signed cubic graph. The following is true:*

(a) $\Phi(G,\sigma) = 3$ *if and only if* $\Phi_c(G,\sigma) = 3$.
(b) *If* $\Phi(G,\sigma) = 4$ *then* $\Phi_c(G,\sigma) = 4$.
(c) *If* $\Phi(G,\sigma) = 5$ *then* $\Phi_c(G,\sigma) \in [4,5]$.

Moreover, there are infinitely many bridgeless signed cubic graphs for every combination of flow and circular flow number satisfying the conditions of the theorem.

Finally, we provide a family of signed graphs for which we conjecture that it is the complete family of signed graphs with flow number 6. We show that all graphs in this family have also the circular flow number 6.

2. Proof of Theorem 1

Switching at a vertex v means reversing the signs of all edges incident with v, except for the loops. Two signed graphs are *switching equivalent* if one can be obtained from the other by a series of switches. Switching equivalent graphs have the same flow number.

Let (O, f) be a positive circular r-flow on a signed cubic graph (G, σ). By the definition of a flow, there are two types of vertices: with one incoming half-edge or with two. According to this partition we colour the vertices white and black. Such a vertex-colouring is called *r-induced colouring*. Note that switching at a vertex changes its colour.

Let (G, σ) be a signed graph and let H be a subgraph of G. Let $E^+(H)$ and $E^-(H)$ be the set of outgoing and incoming half-edges not contained in H incident with a vertex of H, respectively. A *mandating ℓ-path* is one that is switching equivalent to a positive monochromatic path of length ℓ. The *length* of a path is the number of its edges.

Lemma 1. *If there is an r-induced colouring of (G, σ) containing a mandating ℓ-path, then $r \geq \ell + 3$.*

Proof. Let L be a mandating ℓ-path of (G, σ). Switch L to positive monochromatic path. Without loss of generality assume that the vertices of L are black. Then $|E^-(L)| = 2(\ell + 1) - \ell = \ell + 2$ and $|E^+(L)| = \ell + 1 - \ell = 1$. The result follows. □

Let (G, σ) be a cubic signed graph. It can be easily seen that $3 \leq \Phi_c(G, \sigma) \leq \Phi(G, \sigma)$ and therefore if $\Phi(G, \sigma) = 3$, then $\Phi_c(G, \sigma) = 3$.

Now assume that $\Phi_c(G, \sigma) = 3$. Let c be a 3-induced colouring. Since there are no mandating 1-paths, $c(u) = c(v)$ if and only if uv is negative. By switching at all white vertices we obtain an all-negative graph where every vertex is incident with exactly one introverted edge. A nowhere-zero 3-flow can now be easily defined. This proves (a).

The validity of (b) and (c) follows directly from (a) and the following theorem.

Theorem 2 ([**9**])**.** *Let $t \geq 1$ be an integer and (G, σ) be a signed $(2t+1)$-regular graph. If $\Phi_c(G, \sigma) = r$, then $r = 2 + 1/t$ or $r \geq 2 + 2/(2t - 1)$.*

Obviously, there are infinitely many bridgeless signed cubic graphs with $\Phi(G, \sigma) = \Phi_c(G, \sigma) = 3$ and with $\Phi(G, \sigma) = \Phi_c(G, \sigma) = 4$: it suffices to take positive graphs with these flow numbers. Therefore, the values from parts (a) and (b) can be achieved. We prove that the same is true for values in the part (c).

The following theorem follows from [**4, 6**].

Theorem 3. *For every rational number $r \in (4, 5]$ there are infinitely many cubic graphs with circular flow number $\Phi_c(G) = r$.*

Since $\Phi(G) = \lceil \Phi_c(G) \rceil$ for every unsigned graph G, the graphs from Theorem 3 have flow number 5. To complete the proof we show that there exists an infinite family of cubic graphs with flow number 5 and circular flow number 4.

For $i \in \{1,2\}$ let D_i be a graph obtained from K_4 by deleting an edge. Denote the vertices of D_1 and D_2 as $x_0, \dots, x_3$ and $y_0, \dots, y_3$, respectively, where x_0, x_1, y_0, y_1 have degree 2. Let H_n, where $n > 0$, be the graph obtained from D_1 and D_2 by joining the pairs of vertices x_0, y_0 and x_1, y_1 with a path of length $(n+1)$ whose inner vertices are $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, respectively, and by adding edges $a_i b_i$, where $1 \le i \le n$. All edges of (H_n, σ_n) are positive except for x_2x_3, y_2y_3 and a_1a_2, a_ny_0 if $n \ge 2$. Let $e_i = a_ia_{i+1}$ and $g_i = b_ib_{i+1}$ (see Fig. 2a).

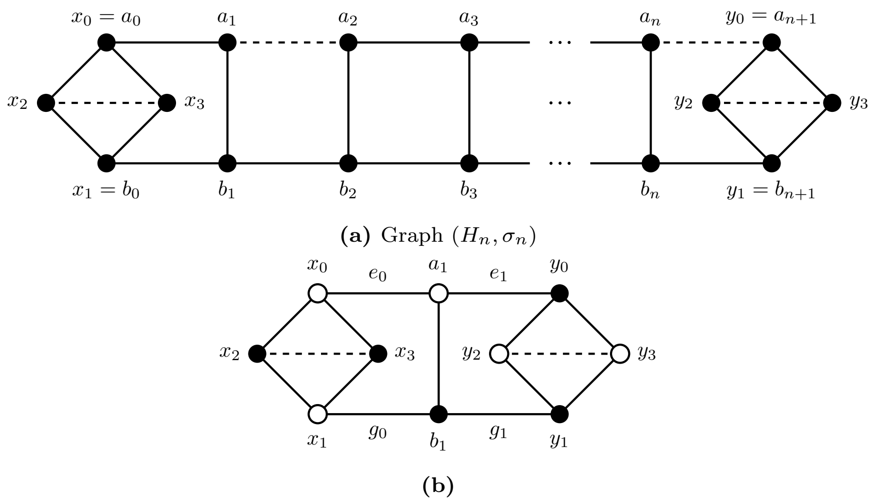

Figure 2

It is not difficult to find a circular 4-flow and a nowhere-zero 5-flow for every (H_n, σ_n). By part (a) of the theorem, it is enough to prove that (H_n, σ_n) does not admit a nowhere-zero 4-flow. We proceed by induction on n.

The induction basis is formed by the graphs with $n \in \{1,2,3\}$. We only prove the nonexistence of a nowhere-zero 4-flow for $n = 1$. Assume that (H_1, σ_1) admits a positive nowhere-zero 4-flow and thus a 4-induced colouring c. By Lemma 1, the coloured graph does not contain a mandating 2-path.

The graph (H_1, σ_1) has two negative edges. The definition of a flow implies that exactly one of them is introverted, therefore the number of incoming half-edges equals the number of outgoing half-edges. It follows that the numbers of black vertices and white vertices coincide. A 4-induced colouring of D_1 has to fulfill following: $c(x_0) = c(x_1)$, $c(x_2) = c(x_3)$ and $c(x_0) \neq c(x_1)$, otherwise there is a mandating 2-path. Similar properties hold for D_2. In each D_i there are two black and two white vertices, therefore $c(a_1) \neq c(b_1)$. Thus all 4-induced colourings are isomorphic to the one in Fig. 2b.

Observe that for every mandating 1-path u, v the flow-value on the edge uv has to be 2 and the flow-values on the other edges incident with u or v have to be 1 or 3. The edges e_0 and g_1 form positive mandating 1-paths in (H_1, σ_1), therefore $f(e_0)+f(g_0)$ is odd. This is not possible since from the definition of a nowhere-zero flow it follows that the sum of flow-values on an edge-cut has to be even.

Assume that $n \geq 4$. Let (O, f) be a nowhere-zero 4-flow on (H_n, σ_n) where every edge $a_i a_{i+1}$, with $1 \leq i < n$, is oriented towards a_{i+1}; similarly every edge $b_i b_{i+1}$, $1 \leq i < n$, is oriented towards b_{i+1}; the edge $a_n y_0$ is oriented from a_n and the edge $b_n y_0$ is oriented from b_n. Since every set $\{e_i, g_i\}$, where $1 \leq i \leq n$, forms an edge-cut, the value $f(e_i) + f(g_i)$ is even and $|f(e_i) + f(g_i)| \leq 6$. Moreover, since the induced subgraph on the vertices $\{a_i, b_i \mid 1 < i \leq n\}$ is positive, the sum $S = f(e_i) + f(g_i)$ is independent of i, where $1 \leq i \leq n$. Therefore, we have seven possible value of S, each of them achievable by at most three different pairs of flow-values.

By the Pigeonhole Principle, there are at least two sets $\{e_i, g_i\}$ and $\{e_j, g_j\}$, $i < j$, with the same pair of values. If we delete the vertices a_k, b_k, where $i < k \leq j$, and all the half-edges incident with them, then there are two pairs of dangling half-edges with the same flow-value. If we join these pairs into edges, we obtain a nowhere-zero 4-flow on a smaller graph, contradicting the minimality of (H_n, σ_n). The proof is complete.

Theorem 1 implies that the conjecture of Raspaud and Zhu is not true even for cubic graphs. Nevertheless, we believe that the following is true.

Conjecture 1. Let (G, σ) be a flow-admissible signed cubic graph. Then

$$\Phi(G, \sigma) - \Phi_c(G, \sigma) \leq 1.$$

3. Signed graphs with flow number 6

Let P^σ denote the Petersen graph with one negative 5-circuit. Let Q^σ be the cube on eight vertices with three independent negative edges, one in each dimension. Let N^σ_{2k+1}, $k \geq 1$, be a signed graph obtained from a positive circuit of length $4k+2$ by replacing every second edge with two parallel edges of different sign (see Fig. 1). These graphs are known to have the flown number 6. We call them the *basic graphs.*

Let (G, σ) and $(H, 1)$ be signed graphs, H bridgeless and positive. Let $u \in G$ and $v \in H$ be vertices of degree 3. Let u_1, u_2 and u_3 and v_1, v_2 and v_3 be the neighbours of u and v, respectively. A signed graph which arises from the union of (G, σ) and $(H, 1)$ by deleting the vertices u and v and joining the vertices u_i and v_i with an edge with sign $\sigma(uu_i)$, where $1 \leq i \leq 3$, is called a *vertex-extension* of (G, σ).

Now we define other type of extension. Let (G, σ) and $(H, 1)$ be signed graphs, H bridgeless. Delete an edge u_1u_2 from (G, σ) and an edge v_1v_2 from $(H, 1)$. Add edges u_1v_1 and u_2v_2. The product of signes of the new edges has to equal to $\sigma(u_1u_2)$. The resulting graph is an *edge-extension* of (G, σ).

A signed graph (G, σ_G) is called a *trivial extension* of a signed graph (H, σ_H) if (H, σ_H) can be transformed to (G, σ_G) by a sequence of vertex- or edge-extensions. Let $\mathcal{M}$ be the set of signed graphs switching equivalent to basic graphs and all their trivial extensions. We propose the following conjecture, which implies Conjecture 1.

Conjecture 2. If a flow-admissible signed graph does not admit a nowhere-zero 5-flow, then it belongs to $\mathcal{M}$.

Theorem 2 and Theorem 1(a) imply that if (G,σ) is cubic and $\Phi(G,\sigma) = 6$, then $\Phi_c(G,\sigma) \in [4,6]$. We prove that the graphs from the family $\mathcal{M}$ have both the flow number and the circular flow number 6. To do so we will use the following lemma.

Lemma 2 ([**3**]). *Let G be a graph and let v be a vertex of G of degree at most 3. If G admits a nowhere-zero k-flow for some integer k, then G admits a nowhere-zero k-flow with arbitrarily prescribed values at the edges around v that sum to 0 and are from the set $\{\pm1, \pm2, \ldots, \pm(k-1)\}$.*

Theorem 4. *Let (G,σ) be a flow-admissible signed graph switching equivalent to a graph from the family $\mathcal{M}$. Then $\Phi(G,\sigma) = \Phi_c(G,\sigma) = 6$.*

Proof. We use induction on the number of extensions. The flow numbers of the basic graphs are equal to 6, see [**1, 5, 9**]. We prove that this is also the case for their circular flow numbers.

To disprove the existence of a circular r-flow on P^σ for $r \le 6$ it can be shown that for every circular r-flow on P^σ there is an $(r-1)$-circular vertex colouring of K_6, a contradiction. In Q^σ, it can be shown that after excluding trivial cases there are eight possibilities of r-induced colourings. In all these cases a mandating 3-path is present, which implies that $r \ge 6$. Finally, if N^σ_{2k+1} admited an r-flow with $r < 6$ for some $k \ge 1$, then the absolute values of the flow-values on the edges with no parallel edge would have to alternate between the intervals $[1,3)$ and $[3,5)$. Since the number of such edges is odd, this is not possible.

If (G,σ) is a trivial extension of (G',σ'), where $(G',\sigma') \in \mathcal{M}$ and $\Phi(G',\sigma') = 6$, then there is a nowhere-zero 6-flow on (G,σ) which is implied by Seymour's 6-flow theorem [**10**] and following Lemma 2.

It remains to prove that extensions do not decrease circular flow number. Assume that (G,σ) was created from (G',σ') by applying a vertex extension with $(H,1)$. Suppose that $\Phi_c(G,\sigma) < 6$. Contract all the edges of (G,σ) corresponding to the edges of H except for the edges incident with the vertex employed in the vertex-extension. Clearly, the resulting graph is (G',σ') and it admits a nowhere-zero r-flow, a contradiction. Edge extensions can be treated similarly. □

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