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PERMUTATION SNARKS OF ORDER 2 (mod 8)

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ABSTRACT. A permutation snark is a cubic graph which has a 2-factor consisting of two chordless cycles and is not 3-edge-colourable. Every permutation snark is cyclically 4-edge-connected, has girth at least 5, and its order is twice an odd number. Employing exhaustive computer search, Brinkmann et al. (2013) discovered a cyclically 5-edge-connected permutation snark of order 34, disproving a conjecture of C.-Q. Zhang (1997) that the Petersen graph is the only such graph. Hägglund and Hoffmann-Ostenhof (2017) extended this example to an infinite series of cyclically 5-edge-connected permutation snarks of order $n = 24k + 10$ for every positive integer k. Here we present three general methods of constructing permutation snarks and with their help provide permutation snarks with cyclic connectivity 4 and 5 for every possible order 2 (mod 8).

1. Introduction

A *cycle permutation graph* is a cubic graph consisting of two disjoint circuits of equal length and a perfect matching between them; equivalently, it is a cubic graph which has a 2-factor consisting of two chordless circuits. A *permutation snark* is a cycle permutation graph that has no 3-edge-colouring.

Cycle permutation graphs were introduced in 1967 by Chartrand and Harary in [**4**] as a generalisation of the Petersen graph and were subsequently studied by various authors, see for example [**8, 11, 12**]. Since the Petersen graph is a snark, permutation snarks are a natural family to investigate. In spite of that, until very recently permutation snarks have attracted only very little attention, with a notable exception of Zhang's book [**13**].

It is not difficult to see that every permutation snark has cyclic connectivity at least 4, girth at least 5, and order twice on odd number. In [**13**], Zhang made a conjecture that the only cyclically 5-edge-connected permutation snark is the Petersen graph. However, in 2013, Brinkmann et al. [**2**] disproved this conjecture by exhibiting a cyclically 5-edge-connected permutation snark on 34

vertices found through an exhaustive computer search. This snark is displayed in Figure 1 in a form rather different from that in [**2**]; the defining 2-factor is shown in bold lines. In 2017, Hägglund and Hoffmann-Ostenhof [**5**] presented an

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ad hoc construction that extends this example to an infinite series of cyclically 5-edge-connected permutation snarks of order $24n + 10$ for each integer $n \geq 1$.

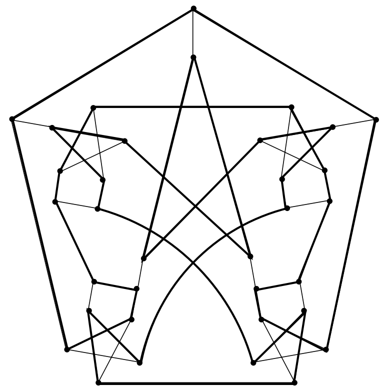

Figure 1. A cyclically 5-connected permutation snark of order 34.

All currently known permutation snarks have order 2 (mod 8) while orders 6 (mod 8) are completely missing. The purpose of this paper to fill in the gaps left by [**5**] and to construct permutation snarks with cyclic connectivity 4 and 5 for every possible order 2 (mod 8). In contrast to [**5**], our constructions are very general and at the same time simple as they only use classical operations: dot product (4-product), a similar operation (5-product) capable of producing cyclically 5-edge-connected snarks, and a subgraph substitution. A brief discussion concerning permutation snarks of order 6 (mod 8) can be found in the last section.

2. Permutation snarks with cyclic connectivity 4

Given two cubic graphs G and H, a *dot product* $G\,.\,H$ is a cubic graph defined as follows. Choose two independent edges $e_1 = a_1b_1$ and $e_2 = a_2b_2$ in G and an edge $e = uv$ in H. Let a_1', b_1', and v be the neighbours of u, and let a_2', b_2', and u be the neighbours of v. Remove the edges e_1 and e_2 from G and the vertices u and v from H. Finally, connect a_1 to a_1', b_1 to b_1', a_2 to a_2', and b_2 to b_2'. Although the notation $G\,.\,H$ is common, the result depends on the choice of the edges e_1 and e_2 in G and the edge $e = uv$ in H as well as on the chosen labelling of the neighbours of u and v in H and their counterparts in G. If we need to be more specific, we will write $G[e_1, e_2]\,.\,[e]H$ instead of $G\,.\,H$.

The operation of dot product was introduced by Adelson-Velskii and Titov [**1**] in 1973 and independently by Isaacs [**6**] in 1975. In [**1**] it was shown that if both G and H are cyclically 4-edge-connected, then so is $G\,.\,H$. Furthermore, $G\,.\,H$ is a snark provided that each of G and H is [**1, 6**]. (Throughout this paper we are using the term *snark* in its most general meaning, that is, as a synonym for a connected cubic graph with no 3-edge-colouring, see e.g. [**3**].)

Now assume that G and H are permutation snarks. Let $P = \{P_1, P_2\}$ be a *permutation* 2-*factor* of G, that is, a 2-factor that consists of two chordless circuits P_1 and P_2. Also let $Q = \{Q_1, Q_2\}$ be a permutation 2-factor of H. Let us perform the dot product $G[e_1, e_2] \,.\, [e]H$ in such a way that the edges e_1 and e_2 belong to different circuits of P and that the edge e is a *spoke* of Q, that is, an edge of the 1-factor complementary to Q. Since the end-vertices vertices u and v of e belong to different circuits of Q, there is an index $i \in \{1, 2\}$ such that the dot product operation welds the circuit P_1 of P with the circuit Q_i of Q and the circuit P_2 of P with Q_{3-i} of Q. In this manner a 2-factor of $G \,.\, H$ consisting of two chordless circuits is produced. Thus $G \,.\, H$ is a permutation snark.

It is well known that the dot product operation is essentially reversible (see for example Cameron et al. [**3**]). This means that if a snark G has a cycle-separating edge-cut S of size 4, then there exist snarks G_1 and G_2 such that G is isomorphic to $G_1 \,.\, G_2$. If G is a permutation snark, then the snarks G_1 and G_2 are uniquely determined by S and both can be shown to be again permutation snarks. Thus the following result is true.

Theorem 2.1. *Let G and H be permutation snarks. A dot-product $G[e_1, e_2] \cdot [e]H$ is a permutation snark if and only if the edges e_1 and e_2 belong to different circuits of a permutation* 2*-factor of G and e belongs to the* 1*-factor complementary to a permutation* 2*-factor of H. Furthermore, every permutation snark with cyclic connectivity* 4 *arises in this way.*

This theorem can be used to produce huge amounts of permutation snarks with cyclic connectivity 4. For example, starting from the Petersen graph and applying Theorem 2.1 repeatedly we can we can construct cyclically 4-edge-connected snarks of every possible order $n \equiv 2 \pmod 8$.

Corollary 2.2. *For every integer $n \equiv 2 \pmod 8$ with $n \geq 10$ there exists a permutation snark of order n.*

Theorem 2.1 explains an explosion of permutation snarks observed by Brinkmann et al. in [**2**]: there is one permutation snark of order 10, two of order 18 (the Blanuša snarks), 64 of order 26, and 10771 of order 34. Exluding the Petersen graph, only twelve of all these snarks are cyclically 5-edge-connected, all of order 34. This naturally directs our interest to cyclically 5-edge-connected permutation snarks.

3. Permutation snarks with cyclic connectivity 5

Dot product has a lesser-known cyclically 5-connected analogue called *star product*. It was introduced by Cameron et al. in [**3**] and can be described as follows. Consider two cubic graphs G and H and containing 5-cycles $C = v_0v_1v_2v_3v_4 \subseteq G$ and $D = w_0w_1w_2w_3w_4 \subseteq H$. For each vertex x on either of these circuits let x' denote the corresponding neighbour not lying on the circuit. Define $G \star H$ to be the cubic graph obtained by removing C from G and D from H and by connecting each vertex v_i' to the vertex w_{2i}' with indices reduced modulo 5. Observe that the result is not uniquely determined as it depends on the chosen labelling of vertices in the 5-cycles C and D. Nevertheless, $G \star H$ is always a snark provided that G and H are (see [**3**]).

Let G be a snark with a permutation 2-factor $P = \{P_1, P_2\}$ and let C be an arbitrary 5-cycle in G. Since each circuit of P is chordless, there exists an index $i \in \{1, 2\}$ such that C has two common edges with P_i and one common edge with P_{3-i}. Hence, precisely one of the edges connecting C to the rest of G is a spoke.

Now let G and H be permutation snarks with 5-cycles $C = v_0v_1v_2v_3v_4 \subseteq G$ and $D = w_0w_1w_2w_3w_4 \subseteq H$. Let us choose the labelling of vertices of the 5-cycles C and D in such a way that v_0v_0' and w_0w_0' are spokes of the respective permutation 2-factors. We will say that the star product performed with respect to such a labelling is *rooted*.

The following result implies that rooted star product can be used to construct permutation snarks.

Proposition 3.1. *A rooted star product of two permutation snarks is again a permutation snark.*

Another method for constructing permutation snarks is based on a subgraph substitution. Take a snark H and construct a subgraph N by removing the vertices of a path R of length 2 from H. The important property of N is that every proper 3-edge-colouring of N induces the same colour on the edges leading from N to one of the end-vertices of R and two different colours on the edges leading to the other end-vertex of R. The fact that N switches matching colours at one end of R to mismatching colours at the other end justifies calling N a *negator*. The edge leading from N to the inner vertex of R is said to be the *residual edge* for N.

Consider a snark G which has a cycle-separating 5-edge-cut S such that one of the components of $G - S$, denoted by M, is isomorphic to the negator N_0 of order 7 obtained from the Petersen graph. Let N be a negator constructed from an arbitrary snark H. Substitute $M \subseteq G$ with N in such a way that the residual edge for M is replaced with the residual edge for N and the pairs of edges determined by the end-vertices of the removed paths are preserved. We say that the resulting graph is obtained from G by a *negator substitution.* It can be shown that a negator substitution performed on a snark gives rise to a snark.

Now assume that both G and H are permutation snarks, G containing a copy M of the Petersen negator. Observe that the 5-edge-cut S that separates M from the rest of G contains a unique spoke. Construct a negator N from H by removing a path $R \subseteq H$ of length 2 that contains a spoke. It follows that the 5-edge-cut between N and R also contains exactly one spoke, but that edge is not residual. A short reflection reveals that if the substitution of $M \subseteq G$ with N is performed in such a way that, additionally, spoke is replaced with a spoke, a permutation snark is obtained. The latter condition still leaves two possibilities how to perform the substitution, and the two graphs obtained in this way will be called *mates.*

Proposition 3.2. *Substituting a Petersen negator in a permutation snark with a negator obtained from a permutation snark can always be performed in such a way that the resulting graph is a permutation snark.*

In order to guarantee a sufficient cyclic connectivity of the resulting permutation snarks we need the following theorem which itself is a consequence of a stronger result independent of edge-colourings.

Theorem 3.3. *Let G and H be cyclically 5-edge-connected permutation snarks. Then:*

(i) *$G \star H$ is cyclically 5-edge-connected;*
(ii) *at least one of the mates obtained by a negator substitution performed on G by using a negator contained in H is cyclically 5-edge-connected.*

We now apply Propositions 3.1 and 3.2 and Theorem 3.3 to construct cyclically 5-edge-connected permutation snarks of every order $n \equiv 2 \pmod 8$ with $n \geq 34$. Note that there exist no cyclically 5-edge-connected permutations of orders strictly between 10 and 34, see [**2**].

For this purpose it is sufficient to display cyclically 5-edge-connected permutation snarks of order 34, 42, and 50, each containing at least two disjoint 5-cycles, and apply rooted star product or a negator substitution repeatedly. A pair of disjoint 5-cycles guarantees that the star product can indeed be iterated, because one of the 5-circuits is used for the product and the other one survives in each factor of the star product.

For the snark of order 34 we can use the snark G_{34} displayed in Figure 1, which clearly contains four disjoint Petersen negators and hence sufficiently many disjoint 5-cycles. To construct a cyclically 5-edge-connected permutation snark G_{42} of order 42 we perform a negator substitution on G_{34} where one Petersen negator is substituted with a negator obtained from the second Blanuša snark B_2 of order 18 (also called Blanuša double according to [**10**]) by removing a path R of length 2 that intersects its unique cycle-separating 4-edge-cut. Note that B_2 is a permutation snark by Theorem 2.1. Although we cannot apply Theorem 3.3 to conclude that G_{42} is cyclically 5-edge-connected, this can be verified directly. A cyclically 5-edge-connected permutation snark G_{50} of order 50 can be obtained by the same procedure applied to G_{42}, since it still contains three disjoint Petersen negators inherited from G_{34}.

Thus we have the following result.

Theorem 3.4. *There exists a cyclically 5-edge-connected permutation snark of order n for each $n \equiv 2 \pmod 8$ with $n \geq 34$.*

4. Final remarks

1. The fact that no permutation snarks of order 6 (mod 8) are currently known is really intriguing. Theorem 3.3 implies that the smallest such snark, if it exists, must be cyclically 5-edge-connected. In [**9**], where full proofs of all our results will appear, we gather additional structural information about a smallest permutation

snark of order 6 (mod 8). In the future, this information could either lead to discovering such a snark or disproving its existence.

2. All known permutation snarks, including those resulting from the constructions described above, have girth 5 and hence cyclic connectivity at most 5. Although cycle permutation graphs of arbitrarily high girth are known [**11**], their cyclic connectivity has not been determined. It is therefore natural to ask whether there exist cycle permutation graphs of arbitrarily high cyclic connectivity. Any effort in this direction could shed more light on the famous conjecture of Jaeger [**7**] that there are no cyclically 7-edge-connected snarks.

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