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# EMBEDDING TREES WITH MAXIMUM AND MINIMUM DEGREE CONDITIONS

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Abstract. We propose the following conjecture: For every fixed $\alpha \in [0, \frac{1}{2}]$, each graph of minimum degree at least $(1+\alpha)\frac{k}{2}$ and maximum degree at least $2(1-\alpha)k$ contains each tree with $k$ edges as a subgraph.
Our main result is an approximate version of the conjecture for bounded degree trees and large dense host graphs. We also show that our conjecture is asymptotically best possible, which disproves a conjecture from [**17**].

## 1. Introduction

A central challenge in extremal graph theory is to determine degree conditions a graph $G$ has to satisfy in order to ensure that it contains a given subgraph $H$. One of the most interesting open cases are trees. Instead of focusing on the containment of just one specific tree $T$, one usually asks for containment of all trees of some fixed size $k \in \mathbb{N}$. To this end, bounds on the average degree, the median degree or the minimum degree of the host graph $G$ have been suggested in the literature. Let us give a quick outline of the most relevant directions.

### 1.1. Average degree

The classical Erdős-Sós conjecture from 1964 (see [**6**]) states that every graph of *average* degree strictly greater than $k-1$ contains each tree with $k$ edges. In particular, this conjecture implies that for every fixed tree $T$ with $k$ edges one has $\mathrm{ex}(n,T) \le \frac{k-1}{2}n$.

This conjecture has received a lot of attention over the last three decades, in particular, Ajtai, Komlós, Simonovits and Szemerédi announced a proof of this conjecture in the early 1990's. Nevertheless, many others partial results have been found since then, see e.g. [**4, 18, 17, 3**].

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### 1.2. Median degree

The Loebl-Komlós–Sós conjecture from 1992 (see [**7**]) states that every graph of *median* degree at least $k$ contains each tree with $k$ edges. For the particular case $k = \frac{n}{2}$, Ajtai, Komlós and Szemerédi [**1**] proved an aproximate version for large $n$, and years later Zhao [**19**] proved the exact version for large $n$.

An approximate version of the Loebl-Komlós-Sós conjecture, for dense graphs, was proved by Piguet and Stein [**14**]. The exact result, for dense graphs, was proved by Piguet and Hladký [**9**], and independently by Cooley [**5**]. For sparse graphs, Hladký, Komlós, Piguet, Szemerédi and Stein proved an approximate version of the Loebl-Komlós-Sós conjecture in a series of four papers [**10, 11, 12, 13**].

### 1.3. Maximum and minimum degree

A new angle to the problem was introduced in 2016 by Havet, Reed, Stein, and Wood [**8**], who impose bounds on both the minimum *and* the maximum degree. They suggest that every graph of minimum degree at least $\lfloor\frac{2k}{3}\rfloor$ and maximum degree at least $k$ contains each tree with $k$ edges. We call their conjecture the $\frac{2}{3}$*–conjecture*, for progress see [**8, 3, 15, 16**].

In [**3**], the present authors proposed a variation of this approach, conjecturing that every graph of minimum degree at least $\frac{k}{2}$ and maximum degree at least $2k$ contains each tree with $k$ edges. We call this conjecture the $2k$–$\frac{k}{2}$ *conjecture.*

### 1.4. New conjecture

Comparing the two variants of maximum/minimum degree conditions given by the latter two conjectures, it seems natural to ask whether one can allow for a wider spectrum of bounds for the maximum and the minimum degree of the host graph. We believe it might be possible to weaken the bound on the maximum degree given by the $2k$–$\frac{k}{2}$ conjecture, if simultaneously, the bound on the minimum degree is increased. Quantitatively speaking, we suggest the following.

**Conjecture 1.1.** Let $k \in \mathbb{N}$, let $\alpha \in [0, \frac{1}{2}]$ and let $G$ be a graph with $\delta(G) \geq (1+\alpha)\frac{k}{2}$ and $\Delta(G) \geq 2(1-\alpha)k$. Then $G$ contains each tree with $k$ edges.

Note that for $\alpha = 0$, the bounds from Conjecture 1.1 coincide with the bounds from the $2k$–$\frac{k}{2}$ conjecture, and for all $\alpha \in [\frac{1}{3}, \frac{1}{2}]$, Conjecture 1.1 follows from the $\frac{2}{3}$–conjecture. We believe that in the range $\alpha \in [\frac{1}{3}, \frac{1}{2}]$, the Conjecture 1.1 is not tight and that the corrects bounds follow from the $\frac{2}{3}$-conjecture (which is stronger in that range).

As positive evidence for Conjecture 1.1, we show an approximate version for trees with bounded degree and dense host graph. Our main result is the following.

**Theorem 1.2.** *For all $\delta \in (0, 1)$ there exist $k_0 \in \mathbb{N}$ such that for all $n, k \geq k_0$ with $n \geq k \geq \delta n$ and for each $\alpha \in [0, \frac{1}{3}]$ the following holds. If $G$ is an $n$-vertex graph with $\delta(G) \geq (1+\delta)(1+\alpha)\frac{k}{2}$ and $\Delta(G) \geq 2(1+\delta)(1-\alpha)k$, then $G$ contains each $k$-edge tree $T$ with $\Delta(T) \leq k^{\frac{1}{67}}$ as a subgraph.*

Due to lack of space we will only give a sketch of the proof of Theorem 1.2, referring to [**2**] for full details. In [**2**] we also discussed the extremal examples and different degree conditions for embedding trees.

## 2. Extremal examples

Let $\ell, k, c \in \mathbb{N}$, with $1 \leq c \leq \frac{k}{\ell(\ell+1)}$, such that $\ell \geq 3$ is odd and divides $k$. For $i = 1, 2$, we define $H_i = (A_i, B_i)$ to be the complete bipartite graph with

$$|A_i| = (\ell - 1)\left(\frac{k}{\ell} - 1\right) \text{ and } |B_i| = \frac{k}{2} + \frac{(c-1)(\ell+1)}{2} - 1.$$

We define the graph $H_{k,\ell,c}$ by adding a new vertex $x$ to $H_1 \cup H_2$, and adding all edges between $x$ and $A_1 \cup A_2$ (see figure 1 below). Observe that

$$\delta(H_{k,\ell,c}) = \min\{|A_1|, |B_1| + 1\} = |B_1| + 1 = \frac{k}{2} + \frac{(c-1)(\ell+1)}{2}$$

and

$$\Delta(H_{k,\ell,c}) = |A_1 \cup A_2| = 2(\ell - 1)\left(\frac{k}{\ell} - 1\right).$$

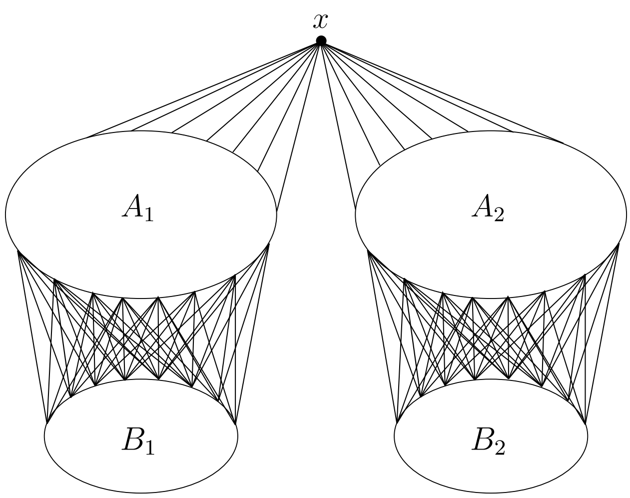



**Figure 1.** The graph $H_{k,\ell,c}$.

The following proposition shows that Conjecture 1.1 is asymptotically tight, we refer to [**2**] for a proof.

**Proposition 2.1.** *For all odd $\ell \in \mathbb{N}$ with $\ell \geq 3$, and for all $\gamma > 0$ there are $k, c \in \mathbb{N}$ and a $k$-edge tree $T$, such that the graph $H_{k,\ell,c}$ satisfies $\delta(H_{k,\ell,c}) \geq (1 + \frac{1}{\ell} - \gamma)\frac{k}{2}$ and $\Delta(H_{k,\ell,c}) \geq 2(1 - \frac{1}{\ell} - \gamma)k$, but $T$ does not embed in $H_{k,\ell,c}$.*

## 3. Sketch of proof of Theorem 1.2

Let $G$ be a graph satisfying the assumptions of Theorem 1.2 and let $x \in V(G)$ be a vertex of maximum degree, that is, $\deg(x) \geq (1+\delta)(1-\alpha)2k$. Our proof uses embedding techniques for trees that the present authors developed in [**3**] and which can be used together with the regularity method.

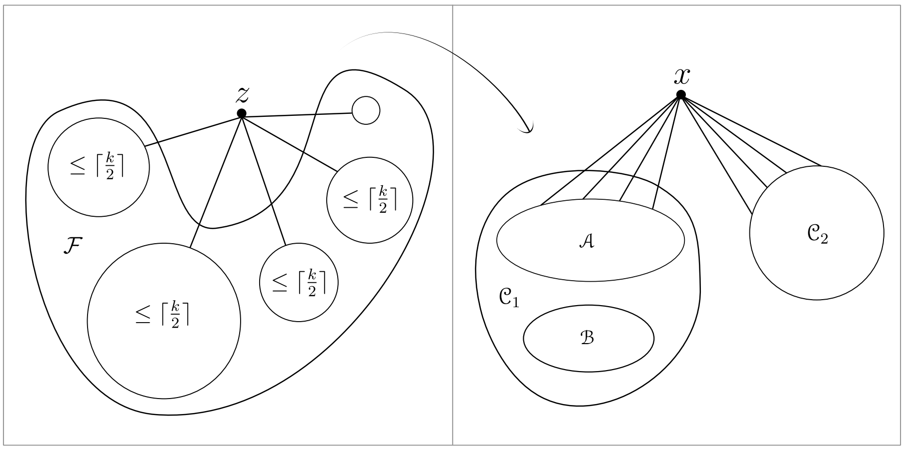
We apply the regularity lemma to $G - x$, with parameters $0 < \varepsilon \ll \eta \ll \delta$, in order to obtain an $(\varepsilon, \eta)$-regular partition of $G - x$ and a corresponding reduced graph $\mathcal{R}$. Given a tree $T$ with $k$ edges and maximum degree at most $k^{\frac{1}{67}}$, we use a general embedding lemma from [**3**] which describes a series of scenarios in which $T$ can be embedded into $G$. If this fails, we deduce that $\mathcal{R}$ has a specific structure that we can use for embedding $T$.

We show that $x$ sees only two components of the reduced graph, say $\mathcal{C}_1$ and $\mathcal{C}_2$[1]. Moreover, the component which receive most of the degree of $x$ is bipartite and $x$ sees only one of the bipartition classes. Assume that $\mathcal{C}_1$ is bipartite with parts $\mathcal{A}$ and $\mathcal{B}$ such that $x$ does not see $\mathcal{B}$, then we have

(I) $(1+\frac{\delta}{2})(1-\alpha)k \leq |V(\bigcup \mathcal{A})| \leq (1+\eta)k$; and

(II) $(1+\frac{\delta}{2})(1+\alpha)\frac{k}{2} \leq |V(\bigcup \mathcal{B})| \leq (1+\eta)k$.

The upper bounds in (I) and (II) follow by the embedding lemma from [**3**]. The lower bound in (I) follows from $\deg(x) \geq (1+\delta)(1-\alpha)2k$ and since $x$ does not see $\mathcal{B}$, the lower bound in (II) is because of the minimum degree of $G$.




[1] We use calligraphic letters to denote subsets of clusters of the reduced graph. Given a component $\mathcal{V}$ of the reduced graph, we write $\bigcup \mathcal{V}$ for the subgraph of $G$ induced by all the clusters in $\mathcal{V}$. Furthermore, we denote by $V(\bigcup \mathcal{V})$ the set of vertices of $G$ that are contained in $\bigcup \mathcal{V}$. Later we will write $\bigcup \mathcal{F}$ for the union of all trees in a family $\mathcal{F}$ of trees.

The plan for embedding $T$ is to split $T$ into a cut vertex and small components that can be arranged into groups that fit into $\mathcal{C}_1$ (respecting its bipartition) and $\mathcal{C}_2$. After that, we embed the cut vertex of $T$ into some special vertex and then we can embed the corresponding forest using the regularity method (the roots of the forest are embedded into neighbours of the image of the cut vertex).

Let $z \in V(T)$ be a vertex such that each component of $T - z$ has size at most $\lceil \frac{k}{2} \rceil$, and let $\mathcal{F}$ denote the set of connected components of $T - z$.

We can partition $\mathcal{F} = \mathcal{J}_1 \cup \mathcal{J}_2$ so that $|V(\mathcal{J}_1)| \le \frac{2}{3}k$ and $|V(\mathcal{J}_2)| \le \frac{k}{2}$. Let $V_0$ be the set of vertices of $T - z$ that lie at even distance to $z$. If $|V_0| \le (1+\alpha)\frac{k}{2}$, we map $z$ into $x$ and. Since

$$|V(\mathcal{J}_1) \cap V_0| \le (1+\alpha)\tfrac{k}{2} \le \frac{1}{1+\frac{\delta}{2}} |V(\textstyle\bigcup \mathcal{B})|$$

and

$$|V(\mathcal{J}_1) \setminus V_0| \le \tfrac{2}{3}k \le (1-\alpha)k \le \frac{1}{1+\frac{\delta}{2}} |V(\textstyle\bigcup \mathcal{A})|,$$

by using regularity we can embed $\mathcal{J}_1$ into $\mathcal{C}_1$, with $V(\mathcal{J}_1) \cap V_0$ going to clusters in $\mathcal{B}$ and $V(\mathcal{J}_1) \setminus V_0$ going to clusters in $\mathcal{A}$. The trees from $\mathcal{J}_2$ can be embedded greedily into $\mathcal{C}_2$ because of the minimum degree of $G$. From now we assume that $|V_0| \ge (1+\alpha)\frac{k}{2}$. The remaining proof splits in two cases.

**Case 1:** $|V(F) \cap V_0| \le \alpha k$ for all $F \in \mathcal{F}$.

Let $\mathcal{F}_1$ be an inclusion-maximal subset $\mathcal{F}_1 \subset \mathcal{F}$ such that

$$|V_0 \cap V(\textstyle\bigcup \mathcal{F}_1)| \le (1+\alpha)\tfrac{k}{2}.$$

Then clearly

$$|V_0 \cap V(\textstyle\bigcup \mathcal{F}_1)| \ge (1-\alpha)\dfrac{k}{2} \qquad \text{and} \qquad |V(\textstyle\bigcup \mathcal{F}_1) \setminus V_0| \le (1+\alpha)\dfrac{k}{2}.$$

This implies that we can embed $z$ into $x$ and thus, by using the regularity lemma, we embed $\mathcal{F}_1$ into $\mathcal{C}_1$, with $V(\bigcup \mathcal{F}_1) \setminus V_0$ going to clusters in $\mathcal{A}$ and $V(\bigcup \mathcal{F}_1) \cap V_0$ going to clusters in $\mathcal{B}$. The trees from $\mathcal{F} \setminus \mathcal{F}_1$ can be embedded greedily into $\mathcal{C}_2$ because of the minimum degree of $G$.

**Case 2:** There is a tree $F^*$ such that $|V(F^*) \cap V_0| > \alpha k$.

Let $\mathcal{F}' = \mathcal{F} \setminus \{F^*\}$ and note that $|V(\bigcup \mathcal{F}') \cap V_0| \le (1-\alpha)k$. This implies that $z$ and $V(\bigcup \mathcal{F}') \cap V_0$ fit into $\mathcal{A}$. Furthermore, since $|V_0| \ge (1+\alpha)\frac{k}{2}$ we have that $|V(\bigcup \mathcal{F}') \setminus V_0| \le (1-\alpha)\frac{k}{2}$. Therefore, $\{z\} \cup (V(\bigcup \mathcal{F}') \cap V_0)$ and $V(\bigcup \mathcal{F}') \setminus V_0$ fit into $\mathcal{A}$ and $\mathcal{B}$ respectively. We embed $z$ into some neighbour of $x$ in $V(\bigcup \mathcal{A})$ and then, using regularity, we can embed $\mathcal{F}'$ into $\mathcal{C}_1$, with $V(\bigcup \mathcal{F}') \cap V_0$ going to clusters in $\mathcal{A}$ and $V(\bigcup \mathcal{F}') \setminus V_0$ going to clusters in $\mathcal{B}$. We embed the root of $F^*$ into $z$ and then complete the embedding into $\mathcal{C}_2$ using the minimum degree of $G$.

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