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SPECTRA AND EIGENSPACES OF ARBITRARY LIFTS OF GRAPHS

C. DALFÓ, M. A. FIOL, S. PAVLÍKOVÁ and J. ŠIRÁŇ

Abstract. We describe, in a very explicit way, a method for determining the spectra and bases of all the corresponding eigenspaces of arbitrary lifts of graphs (regular or not).

1. Introduction

For a graph Γ with adjacency matrix A, the *spectrum* of Γ is defined to be the spectrum of A. While the structure of a graph determines its spectrum, an important converse question in spectral graph theory is to what extent the spectrum determines the structure of a graph; see for example the classical textbooks of Biggs [**2**], and Cvetković, Doob, and Sachs [**3**]. In particular, a persisting problem is to show whether or not a given graph is completely determined by its spectrum (see Van Dam and Haemers [**6**])

Considerable effort has thus been devoted to (partial or entire) identification of spectra of some interesting families of graphs. For instance, Godsil and Hensel [**8**] explicitly studied the problem of determining the spectrum of certain voltage graphs. Lovász [**10**] provided a formula that expresses the eigenvalues of a graph admitting a transitive group of automorphisms in terms of group characters. In the particular case of Cayley graphs (when the automorphism group contains a subgroup acting regularly on vertices), Babai [**1**] derived a more handy formula by different methods (and, as a corollary, obtained the existence of arbitrarily many Cayley graphs with the same spectrum). In fact, Babai's formula also applies to digraphs and arc-colored Cayley graphs. Following these findings, Dalfó, Fiol, and Širáň [**4**] considered a more general construction and derived a method for determining the spectrum of a regular lift of a 'base' (di)graph equipped with an ordinary voltage assignment, or, equivalently, the spectrum of a regular cover of a (di)graph. Recall, however, that by far not all coverings are regular; a description of arbitrary graph coverings by the so-called permutation voltage assignments was given by Gross and Tucker [**9**].

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In this paper we generalize our previous results to arbitrary lifts of graphs (regular or not). Our method not only gives complete spectra of lifts but provides also bases of the corresponding eigenspaces, both in a very explicit way.

Of course, as a consequence, our method also furnishes the associated characteristic polynomials. With respect to the work previously done in this direction, we highlight the work of Feng, Kwak, and Lee [**7**] who solved also the case of graph coverings that are not necessarily regular. In this (and other related) paper, the characteristic polynomials are essentially given in terms of 'large' matrices. Our approach, in contrast, uses relatively 'small' quotient-like matrices derived from the base graph to completely determine the spectrum of the lift (and eigenspace bases).

2. Lifts and permutation voltage assignments

Let Γ be an undirected graph (possibly with loops and multiple edges) and let n be a positive integer. As usual in algebraic and topological graph theory, we will think of every undirected edge joining vertices u and v (not excluding the case when $u = v$) as consisting of a pair of oppositely directed arcs; if one of them is denoted a, then the opposite one will be denoted a^-. Let $V = V(\Gamma)$ and $X = X(\Gamma)$ be the sets of vertices and arcs of Γ. Let G be a subgroup of the symmetric group $\mathrm{Sym}(n)$, that is, a permutation group on the set $[n] = \{1, 2, \ldots, n\}$. A *permutation voltage assignment* on the graph Γ is a mapping $\alpha\colon X \to G$ with the property that $\alpha(a^-) = (\alpha(a))^{-1}$ for every arc $a \in X$. Thus, a permutation voltage assignment allocates a permutation of n letters to each arc of the graph, in such a way that a pair of mutually reverse arcs forming an undirected edge receives mutually inverse permutations.

The graph Γ together with the permutation voltage assignment α determine a new graph Γ^α, called the *lift* of Γ, which is defined as follows. The vertex set V^α and the arc set X^α of the lift are simply the Cartesian products $V \times [n]$ and $X \times [n]$. As regards incidence in the lift, for every arc $a \in X$ from a vertex u to a vertex v for $u, v \in V$ (possibly, $u = v$) in Γ and for every $i \in [n]$ there is an arc $(a, i) \in X^\alpha$ from the vertex $(u, i) \in V^\alpha$ to the vertex $(v, i\alpha(a)) \in V^\alpha$. Note that we write the argument of a permutation to the left of the symbol of the permutation, with composition being read from the left to the right.

If a and a^- are a pair of mutually reverse arcs forming an undirected edge of G, then for every $i \in [n]$ the pair (a, i) and $(a^-, i\alpha(a))$ form an undirected edge of the lift Γ^α, making the lift an undirected graph in a natural way.

The mapping $\pi\colon \Gamma^\alpha \to \Gamma$ that is defined by erasing the second coordinate, that is, $\pi(u, i) = u$ and $\pi(a, i) = a$, for every $u \in V$, $a \in X$ and $i \in [n]$, is a *covering*, in its usual meaning in algebraic topology. Since π consists of n-to-1 mappings $V^\alpha \to V$ and $X^\alpha \to D$, we speak about an *n-fold covering*; we note that this covering may not be regular, see for instance Gross and Tucker [**9**]. The graph Γ is often called the *base graph* of the covering. If Γ is connected, then the lift Γ^α is connected if and only if the voltage group G is a *transitive* permutation group on the set $[n]$. Conversely, given an arbitrary n-fold covering ϑ (regular or not) of our base graph Γ by some graph $\widetilde{\Gamma}$ with vertex set $\widetilde{V}$ and arc set $\widetilde{X}$, then ϑ is equivalent to a covering described above.

Permutation voltage assignments and the corresponding lifts and covers can equivalently be described in the language of the so-called relative voltage assignments as follows. Let Γ be the graph considered above, K a group and L a subgroup of K of index n; we let K/L denote the set of right cosets of L in K. Furthermore, let $\beta\colon X \to K$ be a mapping satisfying $\beta(a^-) = (\beta(a))^{-1}$ for every arc $a \in X$; in this context, one calls β a *voltage assignment in K relative to L*, or simply a relative voltage assignment. The *relative lift* Γ^β has vertex set $V^\beta = V \times K/L$ and arc set $X^\beta = X \times K/L$. Incidence in the lift is given as expected: if a is an arc from a vertex u to a vertex v in Γ, then for every right coset $J \in K/L$ there is an arc (a, J) from the vertex (u, J) to the vertex $(v, J\beta(a))$ in Γ^β. We slightly abuse the notation and denote by the same symbol π the covering $\Gamma^\beta \to \Gamma$ given by suppressing the second coordinate.

There is a one-to-one correspondence between permutation and relative voltage assignments on a connected base graph in the case of *transitive* permutation groups, that is, in the case of a connected lift. Namely, if α is a permutation voltage assignment as in the first paragraph, taking place in a transitive permutation group $G \le \mathrm{Sym}(n)$, then the corresponding relative voltage group is $K = G$, with a subgroup L being the stabilizer in K of an arbitrary point from the set $[n]$. The corresponding relative assignment β is simply identical to α, but acting by right multiplication on K/L. Conversely, if β is a voltage assignment in a group K relative to a subgroup L of index n in K, then one may identify the set K/L with $[n]$ in an arbitrary way, and then $\alpha(a)$ for $a \in X$ is the permutation of $[n]$ induced (in this identification) by right multiplication on the set of (right) cosets K/L by $\beta(a) \in K$. By connectivity of the lift, the permutation group $G = \langle \alpha(a) : a \in X \rangle$ is then a transitive subgroup of $\mathrm{Sym}(n)$.

We also note that a covering $\Gamma^\beta \to \Gamma$, described in terms of a permutation voltage assignment, is regular if and only if the corresponding group G is a regular permutation group on $[n]$. If the covering is given by a voltage assignment in a group K relative to a subgroup L, then it is equivalent to a regular covering if and only if L is a normal subgroup of K. In such a case, the covering admits a description in terms of ordinary voltage assignment in the factor group K/L and with voltage $L\beta(a)$ assigned to an arc $a \in X$ with original relative voltage $\beta(a)$.

Example 2.1 ([**9**])**.** Let Γ be the dumbbell graph with vertex set $V = \{u, v\}$ and arc set $X = \{a, a^-, b, b^-, c, c^-\}$, where the pairs $\{a, a^-\}$, $\{b, b^-\}$ and $\{c, c^-\}$ correspond, respectively, to a loop at u, an edge joining u to v, and a loop at v. Let $n = 3$ and let α be a voltage assignment on Γ with values in $\mathrm{Sym}(3)$ given by $\alpha(a) = (23)$, $\alpha(b) = e$ (the identity element), and $\alpha(c) = (12)$, so that the voltage group G is equal to $\langle (12), (23) \rangle = \mathrm{Sym}(3)$. Since G is transitive and Γ is connected, we may describe the situation in terms of a relative voltage assignment. To do so, we let K be equal to G, and for L we take the subgroup $H = \mathrm{Stab}_G(1) = \{e, (23)\}$ of G. Furthermore, let the set $[3] = \{1, 2, 3\}$ be identified with G/H by $1 \mapsto H$,

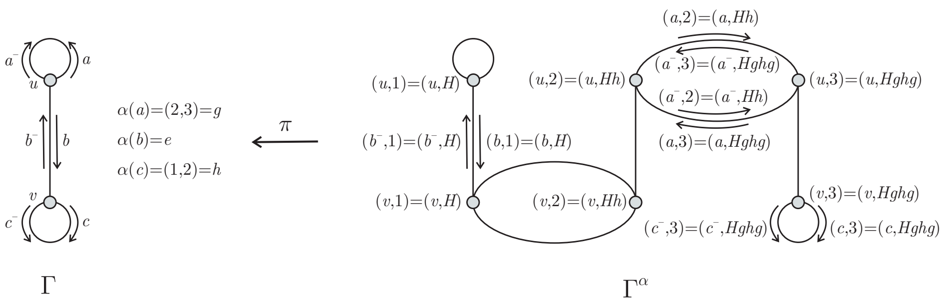

Figure 1. The dumbbell graph Γ and its relative lift Γ^α.

$2 \mapsto H(12)$, and $3 \mapsto H(13)$, and let α be as above but acting by a right multiplication on the right cosets of H in G. Then, for example, in the lift Γ^α the arc $(a, 2)$ emanates from the vertex $(u, 2)$ and terminates at the vertex $(u, 2\alpha(a)) = (u, 3)$ in the permutation voltage setting. Equivalently, in the relative voltage setting, with 2 and 3 identified with $H(12)$ and $H(13)$, the corresponding arc $(a, H(12))$ starting at the vertex $(u, H(12))$ points at the vertex $(u, H(12)\alpha(a)) = (u, H(13))$ because $H(12)\alpha(a) = H(12)(23) = \{(13), (132)\} = H(13)$. The covering $\pi\colon \Gamma^\alpha \to \Gamma$ is irregular, since H is not a normal subgroup of G; the situation is displayed in Figure 1.

In our study we use results from representation theory. For a complex representation ρ of a group G, we let $d_\rho = \dim(\rho)$ denote the dimension of ρ. Furthermore, let $\mathrm{Irep}(G)$ denote a complete set of unitary irreducible representations of G. In the proof of our main result in Section 3 we will also need the following proposition, which we could not locate in the literature and which it could of interest on its own. For a subgroup H of a finite group G and for any $\rho \in \mathrm{Irep}(G)$ we let $\rho(H) = \sum_{h\in H} \rho(h)$.

Proposition 2.2. *For every group G and every subgroup H of G of index n,*

$$\sum_{\rho \in \mathrm{Irep}(G)} \dim(\rho) \cdot \mathrm{rank}(\rho(H)) = n.$$

3. The spectrum of a relative lift

Let Γ be a connected graph on k vertices (again, loops and multiple edges are allowed), and α a permutation voltage assignment on the arc set X of Γ in a *transitive* permutation group G of degree n. Letting, without loss of generality, $H = \mathrm{Stab}_G(1)$, we will be freely using the fact that this assignment is equivalent to the assignment in G relative to H, given by the same mapping α, but considered to act on right cosets of G/H by right multiplication.

To the pair (Γ, α) as above, we assign the $k \times k$ *base matrix* B, a square matrix whose rows and columns are indexed by elements of the vertex set of Γ, and whose uv-th element B_{uv} is determined as follows: If $a_1, \ldots, a_j$ is the set of all the arcs of Γ emanating from u and terminating at v (not excluding the case $u = v$), then

$$B_{u,v} = \alpha(a_1) + \cdots + \alpha(a_j) \ ,$$

the sum being considered to be an element of the (complex) group algebra $\mathbb{C}(G)$; otherwise we let $B_{u,v} = 0$.

As before, let $\rho \in \mathrm{Irep}(G)$ be a unitary irreducible representation of G of dimension $d_\rho = \dim(\rho)$. For our graph Γ on k vertices, the assignment α in G relative to H, and the base matrix B, we let $\rho(B)$ be the $d_\rho k \times d_\rho k$ matrix obtained from B by replacing every non-zero entry $B_{u,v} = \alpha(a_1) + \cdots + \alpha(a_j) \in \mathbb{C}(G)$ as above by the $d_\rho \times d_\rho$ matrix $\rho(B_{u,v})$ defined by

$$\rho(B_{u,v}) = \rho(\alpha(a_1)) + \cdots + \rho(\alpha(a_j)),$$

and by replacing zero entries of B by all-zero $d_\rho \times d_\rho$ matrices. We will refer to $\rho(B)$ as the *ρ-image* of the base matrix B.

An important role in our consideration will be played by the matrix $\rho(H) = \sum_{h\in H} \rho(h)$, that we have encountered in Proposition 2.2; let $r_{\rho,H} = \mathrm{rank}(\rho(H))$. With this notation, we return back to the ρ-image of the base matrix and we let $\mathrm{Sp}(\rho(B))$ denote the spectrum of $\rho(B)$, that is, the multiset of all the $d_\rho k$ eigenvalues of the matrix $\rho(B)$. Finally, let $r_{\rho,H} \cdot \mathrm{Sp}(\rho(B))$ be the multiset of $d_\rho k r_{\rho,H}$ values obtained by taking each of the $d_\rho k$ entries of the spectrum $\mathrm{Sp}(\rho(B))$ exactly $r_{\rho,H} = \mathrm{rank}(\rho(H))$ times. This, in particular, means that if $r_{\rho,H} = \mathrm{rank}(\rho(H)) = 0$, the set $r_{\rho,H} \cdot \mathrm{Sp}(\rho(B))$ is empty. With this terminology and notation in hand, we are ready to state our main result, whose proof can be found in [**5**].

Theorem 3.1. *Let Γ be a base graph of order k and let α be a voltage assignment on Γ in a group G relative to a subgroup H of index n in G. Then, the multiset of the kn eigenvalues of the relative lift Γ^α is obtained by concatenation of the multisets* $\mathrm{rank}(\rho(H))\cdot\mathrm{Sp}(\rho(B))$, *ranging over all the irreducible representations $\rho \in \mathrm{Irep}(G)$.*

In [**5**] it is also shown that all the eigenspaces corresponding to the permutation lift Γ^α are generated by a subset of kn independent columns of a certain product matrix $S^H T$ (see the example of the next section).

4. An example

By way of example for Theorem 3.1, we will now work out the spectrum of the relative lift from Example 2.1. Referring to Figure 1, recall that we consider the dumbbell graph Γ with vertex set $V = \{u, v\}$ and arc set $X = \{a, a^-, b, b^-, c, c^-\}$, where the pairs $\{a, a^-\}$, $\{b, b^-\}$ and $\{c, c^-\}$ determined a loop at u, an edge joining u to v, and a loop at v, respectively. The permutation voltage assignment α on Γ in the group $\mathrm{Sym}(3)$ was given by $\alpha(a) = (23)$, $\alpha(b) = e$, and $\alpha(c) = (12)$. An equivalent description is to regard α as a relative voltage assignment, with values of α on arcs acting on the right cosets of G/H for $H = \mathrm{Stab}_G(1) = \{e, (23)\}$ by right multiplication. Letting $g = (23)$ and $h = (12)$, the base matrix B of G with entries in the group algebra $\mathbb{C}(G)$ has the form

$$B = \begin{pmatrix} \alpha(a)+\alpha(a^-) & \alpha(b) \\ \alpha(b^-) & \alpha(c)+\alpha(c^-) \end{pmatrix} = \begin{pmatrix} 2g & e \\ e & 2h \end{pmatrix}.$$

The (transitive) voltage group $G = \mathrm{Sym}(3) = \{e,g,h,r,s,t\}$ with $r = ghg = (13)$, $s = gh = (123)$ and $t = hg = (132)$ has a complete set of irreducible unitary representations $\mathrm{Irep}(G) = \{\iota, \pi, \sigma\}$ corresponding to the symmetries of an equilateral triangle with vertices positioned at the complex points $\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$, 1, and $\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}$, where

$\iota\colon G \to \{1\}$, $\dim(\iota) = 1$ (the trivial representation),

$\pi\colon G \to \{\pm 1\}$, $\dim(\pi) = 1$ (the parity permutation representation), and,

$\sigma\colon G \to GL(2,\mathbb{C})$, $\dim(\sigma) = 2$, generated by the unitary matrices

$$\sigma(g) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad \text{and} \qquad \sigma(h) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

whence we obtain

$$\sigma(r) = \sigma(ghg) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma(s) = \sigma(gh) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \text{and}$$

$$\sigma(t) = \sigma(hg) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

Then, the images of B under these three representations are given by

$$\iota(B) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ \pi(B) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \ \sigma(B) = \begin{pmatrix} -1 & -\sqrt{3} & 1 & 0 \\ -\sqrt{3} & 1 & 0 & 1 \\ 1 & 0 & -1 & \sqrt{3} \\ 0 & 1 & \sqrt{3} & 1 \end{pmatrix},$$

with spectra $\mathrm{Sp}(\iota(B)) = \{1,3\}$, $\mathrm{Sp}(\pi(B)) = \{-3,-1\}$, and $\mathrm{Sp}(\sigma(B)) = \{\pm\sqrt{3}, \pm\sqrt{7}\}$. To determine the 'multiplication factors' appearing in front of the spectra in the statement of Theorem 3.1, we evaluate $\iota(H) = \iota(e)+\iota(g) = 1+1 = 2$, of rank 1, $\pi(H) = \pi(e)+\pi(g) = 1-1 = 0$, of rank 0, and

$$\sigma(H) = \sigma(e)+\sigma(g) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix},$$

of rank 1. By Theorem 3.1, the spectrum of the adjacency matrix A^α of the relative lift Γ^α is obtained by concatenating the sets $1 \cdot \{1,3\}$, $0 \cdot \{-3,-1\}$, and $1 \cdot \{\pm\sqrt{3}, \pm\sqrt{7}\}$, giving

$$\mathrm{Sp}(A^\alpha) = \{1, \pm\sqrt{3}, \pm\sqrt{7}, 3\}.$$

For simplicity, we now determine the corresponding eigenspaces for only the regular case, that is, when H is the trivial group. The eigenvectors of the ordinary lift Γ_0^α, shown in Figure 2, are obtained from the matrix product ST, where $S = S^H$ is a 12×12 matrix with block form

$$S = (S_\iota \,|\, S_\pi \,|\, S_\sigma) = \left(\begin{array}{cc|cc|cccc} S_{\iota,1} & O & S_{\pi,1} & O & S_{\sigma,1} & O & S_{\sigma,2} & O \\ O & S_{\iota,1} & O & S_{\pi,1} & O & S_{\sigma,1} & O & S_{\sigma,2} \end{array}\right),$$

where

$$S_{\iota,1} = (\iota(e), \iota(g), \iota(h), \iota(r), \iota(s), \iota(t))^\top = (1,1,1,1,1,1)^\top,$$
$$S_{\pi,1} = (\pi(e), \pi(g), \pi(h), \ldots, \pi(t))^\top = (1,-1,-1,-1,1,1)^\top,$$
$$S_{\sigma,1} = (\sigma(e)_1, \ldots, \sigma(t)_1)^\top = \begin{pmatrix} 1 & -1/2 & -1/2 & 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 & 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix}^\top,$$
$$S_{\sigma,2} = (\sigma(e)_2, \ldots, \sigma(t)_2)^\top = \begin{pmatrix} 0 & -\sqrt{3}/2 & \sqrt{3}/2 & 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & 1/2 & 1/2 & -1 & -1/2 & -1/2 \end{pmatrix}^\top.$$

Notice that $S_{\sigma,1}$ and $S_{\sigma,2}$ are formed, respectively, out of the first and second rows of the matrices $\sigma(e)$ and $\sigma(g)$, and so on. Moreover, the matrix $T$is a 12×12 matrix with block form $T = \mathrm{diag}(T_\iota, T_\pi, T_\sigma)$, where $T_\iota = U_\iota$, $T_\pi = U_\pi$, and $T_\sigma = \mathrm{diag}(U_\sigma, U_\sigma)$. Here one needs to be careful about indexation of rows and columns to align eigenvectors with the corresponding eigenvalues. In accordance with the proof of Theorem 3.1 (see again [**5**]), for each $\rho \in \{\iota, \pi, \sigma\}$ of dimension d_ρ the $d_\rho \times d_\rho$ matrix U_ρ is formed by a choice of corresponding eigenvectors of $\rho(B)$. To proceed, we chose to list the eigenvalues $\mu_{(w,i)} = \mu_{(w,i)}(\rho)$ for $w \in V = \{u, v\}$, ρ as above, and $i \in [d_\sigma]$, in the form $\mu_{(u,1)}(\iota) = 3$, $\mu_{(v,1)}(\iota) = 1$, $\mu_{(u,1)}(\pi) = -3$, $\mu_{(v,1)}(\pi) = -1$, $\mu_{(u,1)}(\sigma) = \sqrt{3}$, $\mu_{(u,2)}(\sigma) = \sqrt{7}$, $\mu_{(v,1)}(\sigma) = -\sqrt{7}$, and $\mu_{(v,2)}(\sigma) = -\sqrt{3}$, together with a choice of the corresponding eigenvectors as follows:

$$U_\iota = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \; U_\pi = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \; U_\rho = \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{3} & -1 \\ -1 & \sqrt{7}+2 & \sqrt{7}-2 & -1 \\ 1 & \sqrt{3} & -\sqrt{3} & -1 \\ 1 & \sqrt{7}+2 & \sqrt{7}-2 & 1 \end{pmatrix},$$

where, from left to right, the columns of U_ι correspond to eigenvalues 3 and 1, the columns of U_π to the eigenvalues -3 and -1, and finally the columns of U_σ correspond to the eigenvalues $\sqrt{3}, \sqrt{7}, \sqrt{-7}$ and $\sqrt{-3}$, respectively. Then, the product ST yields all the eigenvectors of the adjacency matrix A^α of the regular lift, corresponding to the eigenvalues ± 3, ± 1, $\pm\sqrt{3}$, and $\pm\sqrt{7}$ (the last four with multiplicity 2).

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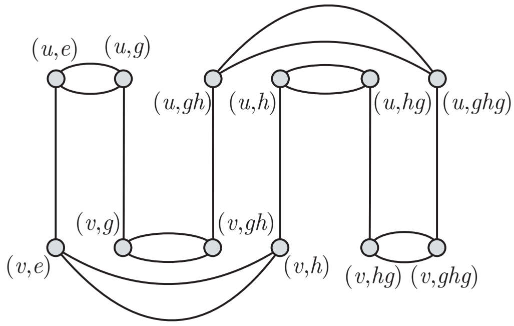

Figure 2. The ordinary (regular) lift Γ_0^α of our base dumbbell graph Γ .

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C. Dalfó, Departament de Matemàtica, Universitat de Lleida, Igualada (Barcelona), Catalonia,
e-mail: cristina.dalfo@matematica.udl.cat

M. A. Fiol, Departament de Matemàtiques, Universitat Politècnica de Catalunya; and Barcelona Graduate School of Mathematics, Barcelona, Catalonia,
e-mail: miguel.angel.fiol@upc.edu

S. Pavlíková, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Bratislava, Slovakia,
e-mail: sona.pavlikova@stuba.sk

J. Širáň, Department of Mathematics and Statistics, The Open University, Milton Keynes, UK; Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Bratislava, Slovakia,
e-mail: j.siran@open.ac.uk