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AN EXPONENTIAL DIOPHANTINE EQUATION RELATED TO ODD PERFECT NUMBERS

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Abstract. We show that for any given primes $\ell \geq 17$ and $p, q \equiv 1 \pmod{\ell}$, the diophantine equation $(x^\ell - 1)/(x - 1) = p^m q$ has at most four positive integral solutions (x, m) and gives its application to odd perfect number problem.

1. Introduction

The purpose of this paper is to bound the number of integral solutions of the diophantine equation

$$\frac{x^\ell - 1}{x - 1} = p^m q, \qquad m \geq 0. \tag{1}$$

This equation arises from our study of odd perfect numbers of a certain form. N is called perfect if the sum of divisors of N except N itself is equal to N. It is one of the oldest problem in mathematics whether or not an odd perfect number exists. Euler has shown that an odd perfect number must be of the form $N = p^\alpha q_1^{2\beta_1} \dots q_k^{2\beta_k}$ for distinct odd primes $p, q_1, \dots, q_k$ and positive integers $\alpha, \beta_1, \dots, \beta_r$ with $p \equiv \alpha \equiv 1 \pmod{4}$.

However, we do not know a proof of the nonexistence of odd perfect numbers even of the special form $N = p^\alpha (q_1 q_2 \dots q_k)^{2\beta}$, although McDaniel and Hagis conjecture that there exists no such one in [**15**]. Gathering various results such as [**4**], [**9**] [**10**], [**11**], [**14**], [**15**], and [**18**], we know that $\beta \geq 9$, $\beta \not\equiv 1 \pmod{3}$, $\beta \not\equiv 2 \pmod{5}$, and β cannot take some other values such as $11, 14, 18, 24$.

We have shown, that if $N = p^\alpha (q_1 q_2 \dots q_k)^{2\beta}$ is an odd perfect number, then $k \leq 4\beta^2 + 2\beta + 2$ in [**19**]. Recently, in [**21**], we have improved this upper bound by $2\beta^2 + 8\beta + 2$ where the coefficient 8 of β can be replaced by 7 if $2\beta + 1$ is not a prime or $\beta \geq 29$. Since it is known that $N < 2^{4^{k+1}}$ from [**17**], we have

$$N < 2^{4^{2\beta^2+8\beta+3}}.$$

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The key point for this result is the diophantine lemma that if ℓ, p, q are given primes such that $\ell \geq 19$ and $p \equiv q \equiv 1 \pmod{\ell}$, then (1) has at most six integral solutions (x, m) such that x is a prime below $2^{4^{\ell^2}}$, and at most five such solutions if ℓ is a prime ≥ 59 (we note that by Theorems 94 and 95 in Nagell [**16**], any prime factor of $(x^\ell - 1)/(x-1)$ with ℓ prime must be $\equiv 1 \pmod{\ell}$ or equal to ℓ). Combining this result with an older upper bound from [**19**], we obtain the above upper bound for N.

Now we return to the equation (1), which is a special type of Thue-Mahler equations. Evertse gave an explicit upper bound for the numbers of solutions of such equations. [**8**, Theorem 3] gives that a slightly generalized equation $(x^\ell - y^\ell)/(x-y) = p^m q^n$ has at most $2 \times 7^{7(\ell-1)^3}$ integral solutions for $\ell \geq 4$. In this paper, we would like to obtain a stronger upper bound for the numbers of solutions of (1).

Theorem 1.1. *If ℓ, p, q are given primes such that $\ell \geq 17$ and $p \equiv q \equiv 1 \pmod{\ell}$, then (1) has at most four positive integral solutions (x, m). Moreover, if p, q, ℓ are such given primes and (1) has five integral solutions (x_i, m_i) with $m_5 > m_4 > \cdots > m_1 \geq 0$, then $m_1 = 0$ and $x_2 = x_1^r$ for some prime $r \neq \ell$.*

Combining this result with an argument in [**21**], we obtain the following new upper bound for odd perfect numbers of a special form.

Corollary 1.2. *If $N = p^e (q_1 q_2 \ldots q_k)^{2\beta}$ is an odd perfect number with $p, q_1, q_2, \ldots, q_k$ distinct primes and $p \equiv e \equiv 1 \pmod 4$ then, $k \leq 2\beta^2 + 6\beta + 2$ and $N < 2^{4^{2\beta^2+6\beta+3}}$.*

Our method is similar to the approach used in [**21**]. In this paper, we use upper bounds for sizes of solutions of (1) derived from a Baker-type estimate for linear forms of logarithms by Matveev [**13**], which may be interesting itself, while [**21**] used an older upper bound for odd perfect numbers of the form given above. We note that Padé approximations using hypergeometric functions given by Beukers [**2**], [**3**], does not work in our situation since our situation gives much weaker approximation to $\sqrt{D}$, although Beukers' gap argument is still useful (see Lemma 2.4 below).

In the next section, we introduce some arithmetic preliminary results from [**21**] and Matveev's lower bound for linear forms of logarithms. In Section 3, using Matveev's lower bound, an upper bound for the sizes of solutions of (1) is given. In Section 4, we prove Theorem 1.1. For large ℓ, this can be done combining results in Sections 2 and 3 with general estimates for class numbers, and regulators of quadratic fields. For small ℓ, we settle the case x_1 is large and then check the remaining x_1's.

A more generalized equation of (1) is

$$\frac{x^\ell - 1}{x - 1} = y^m z^n, \qquad x \geq 2,\ y \geq 2,\ \ell \geq 3,\ mn \geq 2. \tag{2}$$

Assuming the *abc*-conjecture, the author [**20**] proved that any integral solution of (2) with $\ell \geq 3$, $m \geq 1$, $n \geq 2$, $1 \leq y < z$, and x^ℓ sufficiently large must satisfy $(\ell, m, n) = (4, 1, 2)$, $(3, 1, 3)$ or $(\ell, n) = (3, 2)$.

2. Preliminary lemmas

In this section, we introduce some notations and lemmas.

We begin by introducing a well-known result concerning prime factors of values of the n-th cyclotomic polynomial, which we denote by $\Phi_n(X)$. This result was proved by Bang [**1**] and rediscovered by many authors such as Zsigmondy [**22**], Dickson [**7**], and Kanold [**11, 12**].

Lemma 2.1. *If a is an integer greater than 1, then $\Phi_n(a)$ has a prime factor which does not divide $a^m - 1$ for any $m < n$, unless $(a, n) = (2, 6)$ or $n = 2$, and $a + 1$ is a power of 2.*

In order to introduce further results on values of cyclotomic polynomials, we need some notations and results from the arithmetic of a quadratic field. Let $\ell \geq 17$ be a prime and $D = (-1)^{\frac{\ell-1}{2}}\ell$. Let $\mathcal{K}$ and $\mathcal{O}$ denote $\mathbf{Q}(\sqrt{D})$ and its ring of integers $\mathbf{Z}[(1 + \sqrt{D})/2]$, respectively. We use the overline symbol to express the conjugate in $\mathcal{K}$. In the case $D > 0$, ε and $R = \log \varepsilon$ denote the fundamental unit and the regulator in $\mathcal{K}$, respectively. In the case $D < -4$, we set $\varepsilon = -1$ and $R = \pi i$. We note that neither $D = -3$ nor -4 occurs since we have assumed that $\ell \geq 17$.

Moreover, we define the absolute logarithmic height $\mathrm{h}(\alpha)$ of an algebraic number α in $\mathcal{K}$. For an algebraic number α in $\mathcal{K}$ and a prime ideal $\mathfrak{p}$ over $\mathcal{K}$ such that $\alpha = (\zeta_1/\zeta_2)\xi$ with $\xi \in \mathfrak{p}^k$ and ζ_1, ζ_2 in $\mathcal{O} \setminus \mathfrak{p}$, we define the absolute value $|\alpha|_\mathfrak{p}$ by

$$|\alpha|_\mathfrak{p} = N\mathfrak{p}^{-k}$$

as usual, where $N\mathfrak{p}$ denotes the norm of $\mathfrak{p}$, i.e., the rational prime lying over $\mathfrak{p}$. Now the absolute logarithmic height $\mathrm{h}(\alpha)$ is defined by

$$\mathrm{h}(\alpha) = \frac{1}{2}\left(\log^+ |\alpha| + \log^+ |\bar{\alpha}| + \sum_{\mathfrak{p}} \log^+ |\alpha|_\mathfrak{p}\right),$$

where $\log^+ t = \max\{0, \log t\}$ and $\mathfrak{p}$ in the sum runs over all prime ideals over $\mathcal{K}$.

The following three lemmas on the value of the cyclotomic polynomial $\Phi_\ell(x)$ are quoted from [**21**], except the latter part of Lemma 2.3.

Lemma 2.2. *If x is an integer $> \ell^2$, then $\Phi_\ell(x)$ can be written in the form $X^2 - DY^2$ for some coprime integers X and Y with*

$$\left|\frac{Y}{X + Y\sqrt{D}}\right| > \frac{0.4387}{x} \tag{3}$$

and

$$\left|\frac{Y}{X - Y\sqrt{D}}\right| < \frac{0.5608}{x}. \tag{4}$$

Moreover, if p, q *are primes* $\equiv 1 \pmod{\ell}$ *and* $\Phi_\ell(x) = p^m q$ *for some integer* m, *then*

$$\left[\frac{X+Y\sqrt{D}}{X-Y\sqrt{D}}\right] = \left(\frac{\bar{\mathfrak{p}}}{\mathfrak{p}}\right)^{\pm m}\left(\frac{\bar{\mathfrak{q}}}{\mathfrak{q}}\right)^{\pm 1}, \tag{5}$$

where $[p] = \mathfrak{p}\bar{\mathfrak{p}}$ *and* $[q] = \mathfrak{q}\bar{\mathfrak{q}}$ *are prime ideal factorizations in* $\mathcal{O}$.

Lemma 2.3. *Assume that* ℓ *is a prime* ≥ 17. *If* $x_2 > x_1 > 0$ *are two multiplicatively independent integers,* $\Phi_\ell(x_1) = p^{m_1}q$ *and* $\Phi_\ell(x_2) = p^{m_2}q$, *then* $x_2 > x_1^{\lfloor(\ell+1)/6\rfloor}$. *If* $x_2 > x_1 > 0$ *are multiplicatively dependent integers and* $\Phi_\ell(x_i) = p^{m_i}q$ *for* $i = 1, 2$, *then* $m_1 = 0$ *and* $x_2 = x_1^r$ *for some prime* $r \neq \ell$.

Lemma 2.4. *If* $\Phi_\ell(x_i) = p^{m_i}q$ *for three integers* $x_3 > x_2 > x_1 > 0$ *with* $x_2 > x_1^{\lfloor(\ell+1)/6\rfloor}$, *then* $m_3 > 0.445\,|R|\,x_1/\sqrt{\ell}$.

Proofs of lemmas. The former statement of Lemma 2.3 and Lemma 2.4 are 4.1 and 4.2 of [**21**] (the original version of Lemma 4.2 contains an error, see the corrigendum), respectively, for $\ell \geq 19$, and the corresponding statements can be proved for $\ell = 17$ in a similar way. Moreover, Lemma 2.2 is Lemma 2.3 of [**21**] with $3^{\lfloor(\ell+1)/6\rfloor}$ replaced by ℓ^2 for $\ell \geq 19$ and can be proved in a similar way, even for $\ell = 17$. Hence, what we should prove here is only the latter statement of Lemma 2.3.

The assumption implies that $x_1 = y^{r_1}$ and $x_2 = y^{r_2}$ for some positive integers y, r_1, r_2 with $r_2 > r_1$. Assume that $r_1 > 1$, and put $r_i = s_i t_i$ with $t_i = \ell^{k_i}$ and s_i not divisible by ℓ for $i = 1, 2$.

If at least one $s_i \neq 1$, then $\Phi_\ell(y^{r_i})$ must be divisible by $\Phi_{t_i\ell}(y)\Phi_{r_i\ell}(y)$. Hence, three values $\Phi_{t_i\ell}(y)$, $\Phi_{r_1\ell}(y)$, and $\Phi_{r_2\ell}(y)$ must be composed only by p and q. However, since we have assumed that $\ell \geq 17$, Lemma 2.1 yields that each of $\Phi_{t_i\ell}(y)$, $\Phi_{r_1\ell}(y)$, and $\Phi_{r_2\ell}(y)$ must have a primitive prime factor. This is a contradiction.

If $s_1 = s_2 = 1$, then we have $t_1 \neq t_2$ and $\Phi_\ell(x_i) = \Phi_{t_i\ell}(y)$ for $i = 1, 2$. Hence, both $\Phi_{t_1\ell}(y)$ and $\Phi_{t_2\ell}(y)$ must be divisible by q, which is impossible since $q \equiv 1 \pmod{\ell}$.

Thus we must have $r_1 = 1$ and $x_2 = x_1^r$. If r is divisible by ℓ, then, writing $r = s\ell^k$ with s indivisible by ℓ, we see that

$$p^{m_2}q = \Phi_\ell(x_1^r) = \frac{(x_1^{s\ell^{k+1}} - 1)}{(x_1^{s\ell^k} - 1)} = \prod_{d|s} \Phi_{d\ell^{k+1}}(x_1).$$

If $s \neq 1$, then three values $\Phi_\ell(x_1), \Phi_{\ell^{k+1}}(x_1)$, and $\Phi_{s\ell^{k+1}}(x_1)$ must be composed only by p and q, which is impossible like above. Then $s = 1$, and q must divide both $\Phi_\ell(x_1) = p^{m_1}q$ and $\Phi_{\ell^{k+1}}(x_1)$. But this cannot occur since $q \equiv 1 \pmod{\ell}$.

Hence, r is not divisible by ℓ and we see that

$$p^{m_2}q = (x_1^{r\ell} - 1)/(x_1^r - 1) = \prod_{d|r} \Phi_{d\ell}(x_1),$$

while each $\Phi_{d\ell}(x_1)$ has a primitive prime factor. Hence, r must be prime and since $\Phi_\ell(x_1)$ must be divisible by q, we conclude that $\Phi_{r\ell}(x_1) = p^{m_2}$ and $\Phi_\ell(x_1) = q$, proving the latter statement of Lemma 2.3. □

In order to obtain an upper bound for the size of solutions, we use a lower bound for linear forms of logarithms due to Matveev [**13**, Theorem 2.2].

Lemma 2.5. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be algebraic integers in $\mathcal{O}$ which are multiplicatively independent and $b_1, b_2, \dots, b_n$ be arbitrary integers. Let $A(\alpha) = \max\{2h(\alpha), |\log \alpha|\}$ and $A_j = A(\alpha_j)$. Moreover, we put $\kappa = 1$ if $D > 0$, and $\kappa = 2$ if $D < 0$.*

Put

$$
\begin{aligned}
B &= \max\{1, |b_1|\, A_1/A_n, |b_2|\, A_2/A_n, \dots, |b_n|\},\\
\Omega &= A_1 A_2 \dots A_n,\\
C_\kappa(n) &= \frac{16}{n!\kappa} e^n (2n+1+2\kappa)(n+2)(4(n+1))^{n+1}\\
&\quad \times \left(\frac{1}{2}en\right)^\kappa (4.4n + 5.5\log n + 7 + 2\log 2 + \log(1+\log(2))),\\
c &= 3e(1+\log 2),
\end{aligned} \tag{6}
$$

and

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n. \tag{7}$$

Then we have $\Lambda = 0$ or

$$\log |\Lambda| > -C_\kappa(n)(\log cB) \max\left\{1, \frac{n}{6}\right\} \Omega. \tag{8}$$

3. Upper bounds for the sizes of solutions

In this section, we give upper bounds for the sizes of solutions of (1), which itself may be of interest. As in the previous sections, for a prime $\ell \geq 17$, we let $D = (-1)^{\frac{\ell-1}{2}}\ell$, $\mathcal{K}$ and $\mathcal{O}$ denote the quadratic field $\mathbf{Q}(\sqrt{D})$ and its ring of integers $\mathbf{Z}[(1+\sqrt{D})/2]$, respectively, and h is the class number of $\mathcal{K}$. In the case $D > 0$, ε and $R = \log \varepsilon$ denote the fundamental unit and the regulator in $\mathcal{K}$, respectively. In the case $D < -4$, we set $\varepsilon = -1$ and $R = \pi i$. We note that $|R| > \log(\sqrt{17}) > 1.4$ for every D with $|D| \geq 17$.

We let p, q be primes $\equiv 1 \pmod{\ell}$. Then, we can factor $[p] = \mathfrak{p}\bar{\mathfrak{p}}$ and $[q] = \mathfrak{q}\bar{\mathfrak{q}}$ in $\mathcal{O}$ and see that $\mathfrak{p}^h = [\tau]$ and $\mathfrak{q}^h = [\eta]$ for some $\tau, \eta \in \mathcal{O}$. In the case $D > 0$, taking integers u, v so that $|\tau\varepsilon^u| \leq p^{h/2}\varepsilon^{1/2} \leq \left|\tau\varepsilon^{u+1}\right|$ and $|\eta\varepsilon^v| \leq p^{h/2}\varepsilon^{1/2} \leq \left|\eta\varepsilon^{v+1}\right|$, we can take $\tau_0 = \tau\varepsilon^u$, and $\eta_0 = \eta\varepsilon^v$ in $\mathcal{O}$ such that $[\tau_0] = \mathfrak{p}^h, [\eta_0] = \mathfrak{q}^h$ and $p^{h/2}\varepsilon^{-1/2} \leq |\tau_0| \leq p^{h/2}\varepsilon^{1/2}, q^{h/2}\varepsilon^{-1/2} \leq |\eta_0| \leq q^{h/2}\varepsilon^{1/2}$. In the case $D < 0$, we can easily observe that τ_0 and η_0 in $\mathcal{O}$ can be chosen from $\pm\tau, \pm\bar{\tau}$ and $\pm\eta, \pm\bar{\eta}$, respectively, such that $[\tau_0] = \mathfrak{p}^h, [\eta_0] = \mathfrak{q}^h$, and $|\arg \tau_0|, |\arg \eta_0| < \pi/4$.

Theorem 3.1. *Assume that* $\Phi_\ell(x) = p^m q$ *and put* $C = C_1(3) = 1.813\cdots\times 10^{10}$ *if* $\ell \equiv 1 \pmod 4$, *and* $C = C_2(3) = 4.518\cdots\times 10^{10}$ *if* $\ell \equiv 3 \pmod 4$. *Then we have the following upper bounds for* m:

i) *If* $h\log q > h\log p \geq |R|$, *then*

$$m < 4.505C\ell h^2 |R| (\log q)(\log(8cC\ell h^2 |R|) + \log\log p). \tag{9}$$

ii) *If* $h\log q \geq |R| \geq h\log p$, *then*

$$m < 4.505C\frac{\ell}{\log(2\ell)} h |R|^2 (\log q)\log\left(\frac{8cC\ell |R|^3}{\log(2\ell)}\right). \tag{10}$$

iii) *If* $h\log p > h\log q \geq |R|$, *then*

$$m < 4.505C\ell h^2 |R| (\log q)(\log(8cC\ell h^2 |R|) + \log\log q). \tag{11}$$

iv) *If* $h\log p \geq |R| \geq h\log q$, *then*

$$m < 4.505C\ell h |R|^2 \log(8cC\ell h |R|^2). \tag{12}$$

v) *If* $|R| \geq h\log\max\{p,q\}$, *then*

$$m < 4.505C\ell |R|^3 \frac{\log(8cC\ell |R|^3)}{\log\ell}. \tag{13}$$

Proof. We begin by observing that if $m \leq 2\ell\log\ell$, then we can easily confirm the Theorem exploiting the fact that $p, q > 2\ell$. Indeed, in cases i), iii), and iv), we have $2\ell\log\ell < C\ell\log\ell$ which is clearly smaller than the right hand side of the desired inequality in each case. Moreover, in cases ii) and v), we have $|R| \geq \log p > \log(2\ell)$ and $2\ell\log\ell < C\ell\log|R|$ is smaller than the right hand side of the desired inequality in each case. Hence, we may assume that $m > 2\ell\log\ell$, so that $x > \ell^2$. If $\Phi_\ell(x) = p^m q$, then Lemma 2.2 yields that there exist two integers X, Y such that

$$\left[\frac{X+Y\sqrt{D}}{X-Y\sqrt{D}}\right] = \left(\frac{\bar{\mathfrak{p}}}{\mathfrak{p}}\right)^{\pm m}\left(\frac{\bar{\mathfrak{q}}}{\mathfrak{q}}\right)^{\pm 1} \tag{14}$$

with $0 < \left|Y/(X-Y\sqrt{D})\right| < 0.5608/x$. We can easily see that $(X+Y\sqrt{D})/(X-Y\sqrt{D}) \neq \pm 1$ from $Y/(X-Y\sqrt{D}) \neq 0$. Since $|D| = \ell > 3$ is odd, $(X+Y\sqrt{D})/(X-Y\sqrt{D})$ cannot be a root of unity. Hence, taking the h-th powers, we have

$$\left(\frac{X+Y\sqrt{D}}{X-Y\sqrt{D}}\right)^h = \varepsilon^u \left(\frac{\bar{\tau}_0}{\tau_0}\right)^{\pm m}\left(\frac{\bar{\eta}_0}{\eta_0}\right)^{\pm 1} \neq 1 \tag{15}$$

for some integer u, where we take τ_0 and η_0 as we explained just before the lemma. Now let

$$\Lambda = u\log\varepsilon \pm m\log\left(\frac{\bar{\tau}_0}{\tau_0}\right) \pm \log\left(\frac{\bar{\eta}_0}{\eta_0}\right) = h\log\left(\frac{X+Y\sqrt{D}}{X-Y\sqrt{D}}\right). \tag{16}$$

Then, proceeding as in the corrigendum of [**21**], (15) gives that

$$0 < |\Lambda| < \frac{2hY\sqrt{\ell}}{|X - Y\sqrt{D}|} < \frac{1.1216h\sqrt{\ell}}{x}. \tag{17}$$

If $|\Lambda| \geq 1$, then we have $x < 1.1216h\sqrt{\ell} < h\ell$ and $m < \ell \log x/\log p < \ell \log(h\ell)$. We can easily confirm the desired inequality in each case. Hence, we may assume that $|\Lambda| < 1$.

Before applying Lemma 2.5, we must obtain upper bounds for $A(\varepsilon)$, $A(\bar{\tau}_0/\tau_0)$, and $A(\bar{\eta}_0/\eta_0)$. If $D > 0$, then we deduce from $p^{h/2}\varepsilon^{-1/2} \leq |\tau_0| \leq p^{h/2}\varepsilon^{1/2}$ that $|\bar{\tau}_0/\tau_0| \leq \varepsilon$ and $h \log p \leq 2\mathrm{h}(\bar{\tau}_0/\tau_0) \leq h \log p + \log|\varepsilon|$. Thus, we obtain $h \log p \leq A(\bar{\tau}_0/\tau_0) \leq h \log p + |R|$, and similarly, we obtain $h \log q \leq A(\bar{\eta}_0/\eta_0) \leq h \log q + |R|$. Moreover, since $\mathrm{h}(\varepsilon) = (\log \varepsilon)/2$, we have $A(\varepsilon) \leq |R|$. If $D < 0$, then the situation becomes simpler. We can see that $|\tau_0| = |\bar{\tau}_0| = |\eta_0| = |\bar{\eta}_0| = p^{h/2}$. Hence, $\mathrm{h}(\bar{\tau}_0/\tau_0) = \log|\tau_0|_{\mathfrak{p}}^{-1} = (h/2)\log p$, and similarly, $\mathrm{h}(\bar{\eta}_0/\eta_0) = (\log q)/2$. Now, we have $A(\bar{\tau}_0/\tau_0) = \max\{h \log p, \pi/2\} = h \log p$ since $p \geq 47 > e^{\pi/2}$, and, similarly $A(\bar{\eta}_0/\eta_0) = h \log q$. Moreover, $A(\varepsilon) = A(-1) = \pi = |R|$. Thus, in any case, we obtain $h \log p \leq A(\bar{\tau}_0/\tau_0) \leq h \log p + |R|$, $h \log q \leq A(\bar{\eta}_0/\eta_0) \leq h \log q + |R|$, and $A(\varepsilon) \leq |R|$.

We begin by treating the first case $h \log q > h \log p > |R|$. We have

$$\frac{mA(\bar{\tau}_0/\tau_0)}{A(\bar{\eta}_0/\eta_0)} = \frac{m(h \log p + |\log(\bar{\tau}_0/\tau_0)|}{h \log q + |\log(\bar{\eta}_0/\eta_0)|} \leq \frac{m(h \log p + |R|)}{h \log q} \tag{18}$$

and

$$\begin{aligned} \frac{uA(\varepsilon)}{A(\bar{\eta}_0/\eta_0)} = \frac{|u \log \varepsilon|}{A(\bar{\eta}_0/\eta_0)} &\leq \frac{m\,|\log(\bar{\tau}_0/\tau_0)| + |\log(\bar{\eta}_0/\eta_0)| + |\Lambda|}{h \log q} \\ &< \frac{(m+1)\,|R| + |\Lambda|}{h \log q} < \frac{2m\,|R|}{h \log q}, \end{aligned} \tag{19}$$

where we recall that $|\Lambda| < 1$, and observe that $m > 2\ell \log \ell > 48, |R| > 1.4$, and $(m+1)\,|R| + |\Lambda| < (m+1)\,|R| + 1 < 2m\,|R|$.

Since $h \log q > h \log p > |R|$, we see that $A(\bar{\tau}_0/\tau_0) < h \log p + |R| < 2h \log p$, $A(\bar{\eta}_0/\eta_0) < h \log q + |R| < 2h \log q$, and $B \leq 2m \log p/\log q$. Hence, Matveev's theorem gives

$$\begin{aligned} \log x - \log(1.1216h\sqrt{\ell}) &< -\log|\Lambda| \\ &< C(2h)^2 \log\Big(\frac{2cm \log p}{\log q}\Big)\,|R|\,(\log p)(\log q), \end{aligned} \tag{20}$$

and therefore,

$$\begin{aligned} \frac{m \log p}{\log q} &< \frac{\ell \log x}{\log q} \\ &< \ell\Big(\frac{\log(1.1216h\sqrt{\ell})}{\log q} + 4Ch^2\,|R| \log\Big(\frac{2cm \log p}{\log q}\Big)(\log p)\Big). \end{aligned} \tag{21}$$

Taking into account that $C > 10^{10}$, we may assume that $(2cm \log p)/\log q > 10^{10}$, otherwise, (9) automatically holds. Now we observe that $q, p \geq \max\{\ell, 47\}$ and

$2c\log(1.1216h\sqrt{\ell})/\log q < 1+\log h+\log\ell \le 1+\log h+\log p < 2h\log p$. Hence, we obtain

$$\begin{aligned}\frac{2cm\log p}{\log q} &< (8cC+1)\ell h^2\,|R|\log\Big(\frac{2cm\log p}{\log q}\Big)(\log p)\\ &=: U\log\Big(\frac{2cm\log p}{\log q}\Big).\end{aligned}\tag{22}$$

In other words, putting $W = 2cm\log p/\log q$, we have $W/\log W < U$. Since $U > 8cC \ge 8cC_1(3) > 2\times 10^{12}$, we have $(1.12212U\log U)/\log(1.12212U\log U) < U$. Thus we obtain $W < 1.12212U\log U$. Noting that $8cC > 2\times 10^{12}$, $\ell \ge 17$, $|R| > 1.4$, and $p \ge 47$, we have $\log(8cC\ell\,|R|\log p) > 32.84$ and

$$\begin{aligned}\log U &= \log(1.12212(8cC+1)\ell h^2\,|R|\log p)\\ &< \log(8cC\ell h^2\,|R|\log p) + \log(1.12213)\\ &< 1.00351\log(8cC\ell h^2\,|R|\log p).\end{aligned}\tag{23}$$

Hence, (22) yields that

$$\begin{aligned}&\frac{2cm\log p}{\log q} < 1.12212U\log U\\ &< 1.12212(8cC+1)\ell h^2\,|R|\,(\log p)\times 1.00351\log(8cC\ell h^2\,|R|\,(\log p)),\end{aligned}\tag{24}$$

and dividing by $2c$,

$$\frac{m\log p}{\log q} < 4.505C\ell h^2\,|R|\,(\log p)(\log(\ell h^2\,|R|) + \log\log p + \log(8cC)),\tag{25}$$

proving i).

Then, if $h\log q > |R| > h\log p$, then $A(\bar\tau/\tau) < 2\,|R|\,, A(\bar\eta/\eta) < 2h\log q$, and $B \le 2m\,|R|\,/h\log q$. Moreover, (18) and (19) hold as in the previous case. Hence, an argument similar to the above yields that

$$\frac{m\log p}{\log q} < \ell\bigg(\frac{\log(1.1216h)}{\log q} + 4Ch\,|R|^2\log\Big(\frac{2cm\,|R|}{h\log q}\Big)\bigg),\tag{26}$$

and observing that $p > 2\ell$,

$$\frac{m\,|R|}{h\log q} < \ell\bigg(\frac{|R|\log(1.1216h)}{h(\log(2\ell))(\log q)} + 4C\frac{|R|^3}{\log(2\ell)}\log\Big(\frac{2cm\,|R|}{h\log q}\Big)\bigg).\tag{27}$$

We see that $\log(8cC\ell\,|R|^3/\log(2\ell)) > \log(8cC\ell(\log^3 p)/\log(2\ell)) > 33.85$ in this case. Thus, proceeding as above, we obtain

$$\frac{m\,|R|}{h\log q} < 4.505C\frac{\ell}{\log(2\ell)}\,|R|^3\log\Big(\frac{8cC\ell\,|R|^3}{\log(2\ell)}\Big),\tag{28}$$

which proves ii).

In the remaining cases, similar arguments give iii), iv), and v). □

4. Proof of the main theorem

In this section, we prove the main theorem.

Assume that $\Phi_\ell(x_i) = p^{m_i}q$ with $m_i \geq 0$ has five solutions $x_1 < x_2 < x_3 < x_4 < x_5$ such that x_1 and x_2 are multiplicatively independent. It is clear that $x_1 \geq \max\{q^{1/\ell}, 2\}$. Since we have assumed that x_1 and x_2 are multiplicatively independent, Lemma 2.3 yields that $x_3 \geq \max\{q, 2^\ell\}^{\lfloor(\ell+1)/6\rfloor^2/\ell}$. Now it follows from Lemma 2.4 that

$$m_5 > \frac{0.455\pi x_3}{\sqrt{\ell}} > \frac{0.455\pi}{\sqrt{\ell}} \max\{q^{\lfloor(\ell+1)/6\rfloor^2/\ell}, 2^{\lfloor(\ell+1)/6\rfloor^2}\} := M. \tag{29}$$

We begin by the case $\ell \geq 47$. If $\ell \equiv 3 \pmod 4$, then $R = \pi i$. If $\ell \equiv 1 \pmod 4$, then noting that ℓ is prime, it follows from Proposition 3.4.5 of [**5**, p. 138] and Proposition 10.3.16 of [**6**, p. 200] with ℓ in place of m and f in the latter proposition that $hR < \ell^{1/2}((\log \ell)/2 + \log\log \ell + 2.8)$. Now Theorem 3.1 implies that $m_5 < M$, which contradicts to (29). Hence, if $\ell \geq 47$, then $\Phi_\ell(x_i) = p^{m_i}q$ can never have five solutions $x_1 < \cdots < x_5$ such that x_1 and x_2 are pairwise multiplicatively independent.

Next, assume that $\ell = 43$. We must have $x_1 \geq 3$ since $2^{43} - 1 = 431 \times 9719 \times 2099863$ has three distinct prime factors. Thus we must have $m_5 > 0.455\pi \max\{q^{49/43}, 3^{49}\}/\sqrt{43}$, which exceeds the upper bounds given in Theorem 3.1 with $h = 1, R = \pi i$. Indeed, Theorem 3.1 would yield that if $q < 3^{43}$, then $m_5 < 5 \times 10^{16} < 0.455\pi \times 3^{49}/\sqrt{43} < m_5$, and if $q > 3^{43}$, then $m_5 < 2.8 \times 10^{13}(\log q)(\log\log q + 35) < 0.455\pi q^{49/43}/\sqrt{43} < m_5$. In both cases, we are led to a contradiction. Hence, $\Phi_{43}(x_i) = p^{m_i}q$ can never have five solutions $x_1 < \cdots < x_5$ such that x_1 and x_2 are pairwise multiplicatively independent.

Table 1. Estimates when $\ell \leq 41$ and x_1 is large.

ℓ	h	R	$x_1 \geq$	$x_2 >$	$x_3 >$
17	1	$\log(4+\sqrt{17})$	69	x_1^3	$\max\{q^{9/17}, 69^9\}$
19	1	πi	79	x_1^3	$\max\{q^{9/19}, 79^9\}$
23	3	πi	14	x_1^4	$\max\{q^{16/23}, 14^{16}\}$
29	1	$\log((5+\sqrt{29})/2)$	5	x_1^5	$\max\{q^{25/29}, 5^{25}\}$
31	3	πi	6	x_1^5	$\max\{q^{25/31}, 6^{25}\}$
37	1	$\log(6+\sqrt{37})$	3	x_1^6	$\max\{q^{36/37}, 3^{36}\}$
41	1	$\log(32+5\sqrt{41})$	3	x_1^7	$\max\{q^{49/41}, 3^{49}\}$

If $\ell \leq 41$ and x_1 is not less than the corresponding value given in Table 1, then x_2 and x_3 exceed the value given in this table. Now we see that $m_5 > 0.455\pi x_3/\sqrt{\ell}$ exceeds our upper bound M, which leads to a contradiction.

Now we examine the remaining cases. Then $m_1 = 0$ or x_1 must be one of the values given in Table 2, and p, q must be in the range given in this table.

Assume that x_1 is one of the values given in Table 2. In any case, Theorem 3.1 gives that $m < 1.3 \times 10^{17}$. But we have confirmed that $x_2 > p^4 \geq 47^4 > 10^6$ for these cases. Hence, we must have $x_3 > x_2^4 > 10^{24}$ and $m_5 > x_3 > 10^{24}$ for all cases given in Table 2, which is a contradiction again.

Table 2. Estimates when $\ell \leq 41, m_1 > 0$ and x_1 is small.

ℓ	x_1	$p, q \geq$	$p, q \leq$
17	3, 4, 5, 7, 10, 12, 14, 15, 19, 23, 26, 32, 39, 41, 42, 44, 45, 46, 48, 58, 61, 63, 65	103	3627594377435089551046467 59
19	3, 4, 6, 7, 13, 15, 18, 21, 26, 28, 29, 30, 33, 34, 35, 37, 38, 50, 61, 62, 63, 71	191	607127818287731321660577427051
23	2, 3, 5	47	332207361361
23	13	1381	$p_1 = 2519545342349331183143$
31	5	1861	625552508473588471
37	2	223	616318177
41	2	13367	164511353

For example, in the case $\ell = 23$ (in this case, we have $h = 3$ and $R = \pi i$), if $x_1 \geq 14$, then we must have $m_5 > 0.455\pi \max\{q^{16/23}, 14^{16}\}/\sqrt{23}$, which exceeds the upper bounds given in Theorem 3.1.

If $x_1 < 14$, then we must have $x_1 = 2, 3, 5$, or 13, $(10^{23} - 1)/9$ is prime, and $(x^{23} - 1)/(x - 1)$ with $x = 4, 6, 7, 8, 9, 11$, or 12 has more than two distinct prime factors.

If $x_1 = 2, 3$, or 5, then $p, q \leq 332207361361$ and $m < 1.37 \times 10^{17}$. But, in any case, we have confirmed that $x_2 > p^4 > 10^6$. Hence, we must have $x_3 > x_2^4 > 10^{24}$ and $m_5 > x_3 > 10^{24}$, which is a contradiction. If $x_1 = 13$, then $(x_1^{23} - 1)/(x - 1) = 1381p_1$, where $p_1 = 2519545342349331183143$ and $m < 2.46 \times 10^{17}$. Since $(x_1^\ell - 1)/(x_1 - 1) = 1381p_1$, Lemma 2.3 yields that we cannot have $x_2 = x_1^r$ with $r > 1$. Thus, we have $x_2 > 1490855232159367563 99 > 10^{20}$ and $m_4 > 0.455\pi x_2/\sqrt{23} > 2.46 \times 10^{17}$, which is a contradiction.

Next assume that $m_1 = 0$ or equivalently, $(x_1^\ell - 1)/(x_1 - 1) = q$. Thus, $(x, \ell) = (2, 31), (10, 23)$, or $\ell = 19, x \in \{2, 10, 11, 12, 14, 19, 24, 40, 45, 46, 48, 65, 66, 67, 75\}$, or $\ell = 17, x \in \{2, 11, 20, 21, 28, 31, 55, 57, 62\}$. We observe that x_1^r with $1 \leq r \leq \ell - 1$ give the complete solutions to the congruence $(X^\ell - 1)/(X - 1) \equiv 0 \pmod{q}$ and $x_1^{\ell-1} < q$. Since x_1, x_2 are multiplicatively independent, we must have $x_2 > x_1^\ell > \max\{q, 2^\ell\}$. Thus, $x_3 > x_2^{\lfloor(\ell+1)/6\rfloor}$ and $m_5 > 0.455\,|R|\,x_3/\sqrt{\ell}$. However, except the case $\ell = 17$ and $x_1 = 2$, this exceeds the upper bound for m given by Theorem 3.1.

For example, if $\ell = 19$ and $x_1 = 2$, then $x_2 \geq 2^{19}$, $x_3 > x_2^3$, and $m_5 > 0.455\pi x_3/\sqrt{19} > 4.7 \times 10^{16}$ while Theorem 3.1 gives $m_5 < 5.8 \times 10^{15}$.

Finally, if $\ell = 17$ and $x_1 = 2$, then $q = 131071$, and since $(x_2^{17} - 1)/(x_2 - 1) = p^{m_2}q$, we have either $x_2 \geq 2^{18}$ or $x_2 = 131583$. If $x_2 \geq 2^{18}$, then $x_3 > 2^{54}$ and $m_5 > 0.455\log(4 + \sqrt{17})x_3/\sqrt{17} > 4.1 \times 10^{15}$ while Theorem 3.1 gives $m_5 < 1.2 \times 10^{15}$. If $x_2 = 131583$, then $p = 6161\ldots6351$ is a certain 76-digits prime. This yields that $x_3 > 2.3 \times 10^{74}$ and $m_5 > 5.3 \times 10^{73}$, which is impossible again.

Thus, we have proved that if $\Phi_\ell(x_i) = p^{m_i}q$ with $m_i \geq 0$ has five solutions $x_1 < x_2 < x_3 < x_4 < x_5$, then x_1 and x_2 are multiplicatively dependent. Combining it with Lemma 2.3, the proof of Theorem 1.1 is completed.

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