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A MULTIGRID SOLVER FOR CONTROL-CONSTRAINED NAVIER-STOKES CONTROL PROBLEMS

M. M. BUTT

Abstract. A multigrid solver for velocity tracking type control problems constrained by stationary Navier-Stokes equations is presented. Finite difference discretization is used on staggered grids. On these grids, a full multigrid method with coarsening by a factor-of-three strategy is proposed. For smoothing scheme, a distributive Gauss-Seidel scheme is used for the state and adjoint variables and a gradient update step for the control variable is applied. Numerical results validate the efficiency of the proposed multigrid algorithm.

1. Introduction

Optimal control problems constrained by partial differential equations (PDEs) has been an active field of research in applied mathematics (e.g., [**28**]) in the last few decades. In particular, these problems have considerable applications in the field of fluid dynamics. For this reason, we consider one such important field of research, that is, the Navier-Stokes control problem. A lot of works has been done on this topic, e.g., see [**2, 3, 12, 13, 16, 19, 25**]. However, effective solvers for the Navier-Stokes equations are fairly a recent development in Applied Mathematics.

Multigrid methods for the positive definite linear systems arising in elliptic boundary value problems were proven one of the efficient solvers [**6, 17, 18, 30**]. However, for saddle-point systems they are more involved [**1**]. Spatial discretization of the Navier-Stokes equations using either finite element or finite difference method leads to a large sparse saddle point system. A lot of work has been done for developing efficient solvers for the discretized system, especially efficient preconditioners for Krylov subspace methods based on the block matrix form, see [**1, 14**]. Multigrid methods have also been considered for Flow (saddle-point) problems, see [**7, 8, 23, 27**]) and the references therein.

In recent years, many articles have been devoted to developing multigrid methods for optimal control problems, e.g., see Borzì and Schulz [**4**] and the references therein. We are interested in efficient multigrid methods that are robust with

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respect to both the mesh size h and the regularization parameter α without any preconditioner.

In this paper, a multigrid solver on staggered grids using finite differences is developed. In particular, the focus of this paper is more on the numerical efficiency of the proposed multigrid algorithm for solving tracking type control problems. The finite difference discretization applied to the optimality system utilized first-order standard upstream difference scheme (for the convection term) to avoid numerical oscillation. We extend the work given by [**9, 10**], to Navier-Stokes control problems with and without control constraints. Multigrid algorithm with coarsening by a factor of three on staggered grids has the potential advantage of simplifying the intergrid transfer operators, reducing number of levels and thus the computations.

In the next Section 2, tracking type control problem is considered and the solution is characterize as an optimality system. In Section 3, discretization of the optimality system using finite difference on staggered grids is discussed. A multigrid framework is given in Section 4, that consists of a smoothing scheme and a gradient update step for the control variable. In Section 5, we present numerical results for the proposed multigrid scheme. Control-constrained problem with numerical results is given in Section 6, and finally conclusions are given in the last Section.

2. The Navier-Stokes control problem

In this work, we consider the velocity tracking type control problem in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\Gamma = \partial\Omega$: Find a velocity field $\mathbf{u} \in \mathbf{H}^1(\Omega)$, a pressure $p \in L^2(\Omega)$, and a control $\mathbf{f} \in \mathbf{L}^2(\Omega)$ such that the functional

$$J(\mathbf{u}, \mathbf{f}) := \frac{1}{2}\|\mathbf{u} - \mathbf{u_d}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{f}\|^2_{L^2(\Omega)} \tag{1}$$

is a minimized subject to the following stationary Navier-Stokes equations

$$-\nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2}$$

$$-\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{3}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma. \tag{4}$$

The variables $\mathbf{u}$ and p are the state variables, denoting the velocity and pressure, respectively. The (regularization) parameter $\alpha > 0$ represents the weight of the cost of control $\mathbf{f}$, the parameter ν denotes viscosity, and $\mathbf{u_d} \in \mathbf{L}^2(\Omega)$ is the target function (desired state). Furthermore, assume that p satisfies the zero mean constraint, i.e., $\int_\Omega p \, \mathrm{d}x = 0$.

Here and in the following, $L^2(\Omega)$ and $H^1(\Omega)$ denote the standard Lebesque and Sobolev spaces with $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$, respectively, as associated standard norms. The usual inner product associated with $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. Throughout this paper, we follow this same notational convention and use bold script to denote vectors and product spaces. Moreover, we have the space $L^2_0(\Omega)$, which is the space of functions in $L^2(\Omega)$ with mean value 0, i.e.,

$$L^2_0(\Omega) = \left\{ \phi \in L^2(\Omega) : \int_\Omega \phi \,\mathrm{d}x = 0 \right\}$$

and $\mathbf{H}^1_0(\Omega)$, the space in $\mathbf{H}^1(\Omega)$ vanishing on the boundary, i.e.,

$$\mathbf{H}^1_0(\Omega) = \left\{ \psi \in \mathbf{H}^1(\Omega) : \psi = 0 \text{ on } \partial\Omega \right\}.$$

A weak formulation of the Navier-Stokes equations is given as follows. Given $\mathbf{f} \in \mathbf{H}^{-1}$, find $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ of (2)–(4) is the solution of

$$\begin{aligned} a(\mathbf{u},\mathbf{w}) + c(\mathbf{u},\mathbf{u},\mathbf{w}) + b(\mathbf{w},p) &= \langle \mathbf{f},\mathbf{w}\rangle && \text{for all } \mathbf{w} \in \mathbf{H}^1_0(\Omega), \\ b(\mathbf{u},q) &= 0 && \text{for all } q \in L^2_0(\Omega), \end{aligned} \tag{5}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality between $\mathbf{H}^1_0(\Omega)$ and $\mathbf{H}^{-1}(\Omega)$, and

$$\begin{aligned} a(\mathbf{u},\mathbf{w}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{w}) = \nu \sum_{i=1}^{2} \int_\Omega \nabla \mathbf{u}_i \cdot \nabla \mathbf{w}_i & \quad \text{for all } \mathbf{u},\mathbf{w} \in \mathbf{H}^1_0(\Omega), \\ b(\mathbf{w},p) = -\int_\Omega p \,\nabla \cdot \mathbf{w} & \quad \text{for all } \mathbf{w} \in \mathbf{H}^1_0(\Omega) \text{ and } p \in L^2_0(\Omega), \end{aligned}$$

are the bilinear form, and

$$c(\mathbf{u};\mathbf{w},\phi) = ((\mathbf{u}\cdot\nabla)\mathbf{w},\phi) \qquad \text{for all } \mathbf{u},\mathbf{w},\phi \in \mathbf{H}^1_0(\Omega),$$

the trilinear form, respectively.

We recall here the standard result regarding the existence of (5) and uniqueness for small data [**15, 21**] given as follows.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ. Then for every $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, there exists one solution $(\mathbf{u},p) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ of the stationary Navier-Stokes system (5) that satisfies the estimate

$$\|\nabla \mathbf{u}\| \le \nu^{-1} \|\mathbf{f}\|. \tag{6}$$

Moreover, the solution is unique if the data satisfies the smallness condition

$$\mathcal{M}\nu^{-2} < 1, \text{ with } \mathcal{M} = \sup_{\mathbf{u},\mathbf{w},\phi \in \mathbf{H}^1_0(\Omega)\setminus\{0\}} \frac{|c(\mathbf{u};\mathbf{w},\phi)|}{\|\nabla \mathbf{u}\|\|\nabla \mathbf{w}\|\|\nabla \phi\|}. \tag{7}$$

If Ω is a convex polygon and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, then $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$, and

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} \le C(1 + \|\mathbf{f}\|^3).$$

In this paper, we assume Ω to be convex so that the $\mathbf{H}^2$-regularity of the Navier-Stokes system is ensured.

For the purpose of optimal control problem, we now introduce $\lambda \in \mathbf{H}^1_0(\Omega)$, $q \in L^2_0(\Omega)$ as the adjoint variables to $\mathbf{u}$ and p, respectively. The above formulation is well defined. For discussion on existence of the optimal solutions and a derivation of optimality conditions, we refer [**13, 26**] leading to the following optimality system

$$
\begin{aligned}
-\nu\,\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega,\\
-\nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, \quad \text{(state system)},\\
\mathbf{u} &= 0 && \text{on } \Gamma,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
-\nu\,\Delta \lambda - (\mathbf{u}\cdot\nabla)\lambda + (\nabla\mathbf{u})^T\lambda + \nabla q &= \mathbf{u_d} - \mathbf{u} && \text{in } \Omega,\\
\nabla\cdot\lambda &= 0 && \text{in } \Omega, \quad \text{(adjoint system)},\\
\lambda &= 0 && \text{on } \Gamma,
\end{aligned} \tag{9}
$$

$$
\alpha\mathbf{f} - \lambda = 0 \qquad \text{in } \Omega, \quad \text{(optimality conditions)}, \tag{10}
$$

where

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u} &= \left(u^1\partial_x u^1 + u^2\partial_y u^1,\; u^1\partial_x u^2 + u^2\partial_y u^2\right)^T,\\
(\nabla\mathbf{u})^T\lambda &= \left((\partial_x u^1)\lambda^1 + (\partial_x u^2)\lambda^2,\; (\partial_y u^1)\lambda^1 + (\partial_y u^2)\lambda^2\right)^T.
\end{aligned}
$$

This system characterizes the solution $(\mathbf{u}, p, \mathbf{f}) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times \mathbf{L}^2(\Omega)$ of the optimal control problem with Lagrange multipliers $(\lambda, q) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$.

3. Discretization of the optimality system

In this section, we discretize the optimality system by finite difference approximations on staggered grids. Implementation details are given and the advantageous collocation of variables are noted.

We consider a sequence of grids $\{\Omega_h\}_{h>0}$ given by

$$
\Omega_h = \{\mathbf{x}\in\mathbb{R}^2 : x_i = i\,h,\; y_j = j\,h,\quad i,j\in\mathbb{Z}\}\cap\Omega.
$$

We assume that Ω is a rectangular domain and that the values of h are chosen such that the boundaries of Ω coincide with grid lines. On staggered grids, variables may be placed on cell edge-vertical, edge-horizontal, and on cell centers. We denote these sets of grid points with Ω_h^s, $s\in\{ev, eh, c\}$.

For grid functions u^h and v^h defined on the same set Ω_h^s, we introduce the discrete L^2-scalar product

$$
(u^h, v^h)_{L_h^2(\Omega_h^s)} = h^2 \sum_{\mathbf{x}\in\Omega_h^s} u^h(\mathbf{x})\, v^h(\mathbf{x})
$$

with associated norm $\|u^h\|_{L_h^2(\Omega_h^s)} = (u^h, u^h)^{1/2}_{L_h^2(\Omega_h^s)}$. The spaces $L_h^2(\Omega_h^s)$ and $H_h^1(\Omega_h^s)$ consist of the sets of grid functions u_h defined on Ω_h^s endowed with norm $\|u_h\|_{L_h^2(\Omega_h^s)}$ and $\|u_h\|_{H_h^1(\Omega_h^s)}$, respectively. We denote with $\mathcal{U}_h$, $\mathcal{V}_h$ and $\mathcal{P}_h$ the space of the grid functions u_h^1, u_h^2, and p_h.

First, we discretize the state system using finite differences. On staggered grid, u is defined on Ω_h^{ev}, v is defined on Ω_h^{eh}, and p is defined at cell centers Ω_h^c, see

Figure 1. We can write the discrete state system as

$$Q_h \mathbf{u}_h^j + \partial_j^h p_h = \mathbf{f}_h^j \quad \text{at } j\text{-face centers } (j=1,2), \qquad \sum_{i=1}^{2} \partial_i^h \mathbf{u}_h^i = 0 \quad \text{at cell centers,}$$

where $Q_h = -\nu\,\Delta_h + \sum \mathbf{u}_h^i \partial_i^h$ is some difference approximation to $Q = -\nu\,\Delta + \sum \mathbf{u}^i \partial_i$; Δ_h is the usual 5-point approximation, and for the first-order derivatives, we use the second-order central differences. Moreover, for the convection term, we use first-order upwinding scheme. It is also important to note that near the boundary, Δ_h may involve an exterior (ghost) value, which is defined by quadratic extrapolation cf. [**7**].

Here, we consider a set of grid indices that index all grid points in a lexicographic order, i.e., (i,j), $i = 1,\dots,N_x+1$, $j = 1,\dots,N_y+1$, starting from the lowest-left corner. Moreover, the vertices are given by $x_i = (i-1)\,h$ and $y_j = (j-1)\,h$, $v_{i+1/2,j+1/2}$ we mean the discrete counterpart to $v(x_i + h/2, y_j + h/2)$.

In the following, we use the notation given by [**5**], $u_+ := \max(0,u)$, $u_- := \min(0,u)$, and

$$\chi_+^1 := \begin{cases} 1, & \text{if } \max(0, u^1_{i,j+1/2}) > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi_-^1 := \begin{cases} 1, & \text{if } \min(0, u^1_{i,j+1/2}) < 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi_+^2 := \begin{cases} 1, & \text{if } \max(0, u^2_{i+1/2,j}) > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi_-^2 := \begin{cases} 1, & \text{if } \min(0, u^2_{i+1/2,j}) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then by using the first-order upwind scheme for the convection term, which avoids numerical oscilations with a first-order accuracy, we have the discretized state system given by

$$\begin{aligned} &-\nu\left[\frac{u^1_{i-1,j+1/2} + u^1_{i+1,j+1/2} + u^1_{i,j-1/2} + u^1_{i,j+3/2} - 4u^1_{i,j+1/2}}{h^2}\right] \\ &+ (u^1_{i,j+1/2})_+ \frac{u^1_{i,j+1/2} - u^1_{i-1,j+1/2}}{h} + (u^1_{i,j+1/2})_- \frac{u^1_{i+1,j+1/2} - u^1_{i,j+1/2}}{h} \\ &+ (u^2_{i+1/2,j})_+ \frac{u^1_{i,j+1/2} - u^1_{i,j-1/2}}{h} + (u^2_{i+1/2,j})_- \frac{u^1_{i,j+3/2} - u^1_{i,j+1/2}}{h} \\ &+ \frac{p_{i+1/2,j+1/2} - p_{i-1/2,j+1/2}}{h} = f^1_{i,j+1/2} \qquad \text{on } \Omega_h^{ev}, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\nu\left[\frac{u^2_{i-1/2,j} + u^2_{i+3/2,j} + u^2_{i+1/2,j-1} + u^2_{i+1/2,j+1} - 4u^2_{i+1/2,j}}{h^2}\right] \\ &+ (u^1_{i,j+1/2})_+ \frac{u^2_{i+1/2,j} - u^2_{i-1/2,j}}{h} + (u^1_{i,j+1/2})_- \frac{u^2_{i+3/2,j} - u^2_{i+1/2,j}}{h} \\ &+ (u^2_{i+1/2,j})_+ \frac{u^2_{i+1/2,j} - u^2_{i+1/2,j-1}}{h} + (u^2_{i+1/2,j})_- \frac{u^2_{i+1/2,j+1} - u^2_{i+1/2,j}}{h} \\ &+ \frac{p_{i+1/2,j+1/2} - p_{i+1/2,j-1/2}}{h} = f^2_{i+1/2,j} \qquad \text{on } \Omega_h^{eh}, \end{aligned} \tag{12}$$

$$\frac{u^1_{i+1,j+1/2} - u^1_{i,j+1/2}}{h} + \frac{u^2_{i+1/2,j+1} - u^2_{i+1/2,j}}{h} = 0 \qquad \text{on } \Omega_h^c, \tag{13}$$

where the equation (11) is centered at all internal cells Ω_h^{ev}, the equation (12) is centered at all internal cells Ω_h^{eh}, and the continuity equation (13) is centered at all internal cell centers Ω_h^c, respectively, see Fig. 1.

Next, on staggered grids, the optimality conditions $\alpha f^1 - \lambda^1 = 0$, and $\alpha f^2 - \lambda^2 = 0$ give that the adjoint system should be such that u^1, λ^1, and f^1 defined on Ω_h^{ev}; u^2, f^2, and λ^2 defined on Ω_h^{eh}; p and q are defined on Ω_h^c. Therefore, we have the following discretized adjoint system

$$
\begin{aligned}
& -\nu\Big[\frac{\lambda^1_{i-1,j+1/2}+\lambda^1_{i+1,j+1/2}+\lambda^1_{i,j-1/2}+\lambda^1_{i,j+3/2}-4\lambda^1_{i,j+1/2}}{h^2}\Big]\\
& -\chi^1_- u^1_{i,j+1/2}\frac{\lambda^1_{i,j+1/2}-\lambda^1_{i-1,j+1/2}}{h}-\chi^1_+ u^1_{i,j+1/2}\frac{\lambda^1_{i+1,j+1/2}-\lambda^1_{i,j+1/2}}{h}\\
& -\chi^2_- u^2_{i+1/2,j}\frac{\lambda^1_{i,j+1/2}-\lambda^1_{i,j-1/2}}{h}-\chi^1_+ u^2_{i+1/2,j}\frac{\lambda^1_{i,j+3/2}-\lambda^1_{i,j+1/2}}{h}\\
& +(\lambda^1_{i,j+1/2})_+\frac{u^1_{i,j+1/2}-u^1_{i-1,j+1/2}}{h}+(\lambda^1_{i,j+1/2})_-\frac{u^1_{i+1,j+1/2}-u^1_{i,j+1/2}}{h}\\
& +(\lambda^2_{i+1/2,j})_+\frac{u^2_{i+1/2,j}-u^2_{i-3/2,j}}{h}+(\lambda^2_{i+1/2,j})_-\frac{u^2_{i+3/2,j}-u^2_{i+1/2,j}}{h}\\
& +\frac{q_{i+1/2,j+1/2}-q_{i-1/2,j+1/2}}{h}=(u^1_d-u^1)_{i,j+1/2} \qquad \text{on } \Omega_h^{ev},
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
& -\nu\Big[\frac{\lambda^2_{i-1/2,j}+\lambda^2_{i+3/2,j}+\lambda^2_{i+1/2,j-1}+\lambda^2_{i+1/2,j+1}-4\lambda^2_{i+1/2,j}}{h^2}\Big]\\
& -\chi^1_- u^1_{i,j+1/2}\frac{\lambda^2_{i+1/2,j}-\lambda^2_{i-1/2,j}}{h}-\chi^1_+ u^1_{i,j+1/2}\frac{\lambda^2_{i+3/2,j}-\lambda^2_{i+1/2,j}}{h}\\
& -\chi^2_- u^2_{i+1/2,j}\frac{\lambda^2_{i+1/2,j}-\lambda^2_{i+1/2,j-1}}{h}-\chi^2_+ u^2_{i+1/2,j}\frac{\lambda^2_{i+1/2,j+1}-\lambda^2_{i+1/2,j}}{h}\\
& +(\lambda^1_{i,j+1/2})_+\frac{u^1_{i,j+1/2}-u^1_{i,j-1/2}}{h}+(\lambda^1_{i,j+1/2})_-\frac{u^1_{i,j+3/2}-u^1_{i,j+1/2}}{h}\\
& +(\lambda^2_{i+1/2,j})_+\frac{u^2_{i+1/2,j}-u^2_{i+1/2,j-1}}{h}+(\lambda^2_{i+1/2,j})_-\frac{u^2_{i+1/2,j+1}-u^2_{i+1/2,j}}{h}\\
& +\frac{q_{i+1/2,j+1/2}-q_{i+1/2,j-1/2}}{h}=(u^2_d-u^2)_{i+1/2,j} \qquad \text{on } \Omega_h^{eh},
\end{aligned} \tag{15}
$$

$$
\frac{\lambda^1_{i+1,j+1/2}-\lambda^1_{i,j+1/2}}{h}+\frac{\lambda^2_{i+1/2,j+1}-\lambda^2_{i+1/2,j}}{h}=0 \qquad \text{on } \Omega_h^c. \tag{16}
$$

Here, we remark that with our approach, we implement a direct coupling among all the variables (of state, adjoint, and control) without the need of interpolation.

Here we also remark that a special care has to be taken while relaxing the state and adjoint variables (on their respective spatial locations) as some of the variables (in the convection term) are not located on the same spatial location, where the respective equation is relaxed. Therefore, we take interpolation (average of four neighbouring points) for such variables, cf. [**8**].

Summarizing, equations (11)–(16) with the following optimality conditions

$$
\begin{aligned}
&\alpha\, f^1_{i,j+1/2} - \lambda^1_{i,j+1/2} = 0 \qquad && \text{on } \Omega_h^{ev}, \\
&\alpha\, f^2_{i+1/2,j} - \lambda^2_{i+1/2,j} = 0 \qquad && \text{on } \Omega_h^{eh},
\end{aligned}
\tag{17}
$$

and boundary conditions

$$
\begin{aligned}
u^1_{i,j+1/2} &= 0 \qquad && \text{for } i = 1,\ N_x+1,\ j = 1, \ldots, N_y, \\
u^2_{i+1/2,j} &= 0 \qquad && \text{for } j = 1,\ N_y+1,\ i = 1, \ldots, N_x, \\
\lambda^1_{i,j+1/2} &= 0 \qquad && \text{for } i = 1,\ N_x+1,\ j = 1, \ldots, N_y, \\
\lambda^2_{i+1/2,j} &= 0 \qquad && \text{for } j = 1,\ N_y+1,\ i = 1, \ldots, N_x,
\end{aligned}
\tag{18}
$$

constitute the discrete optimality system for the control problem.

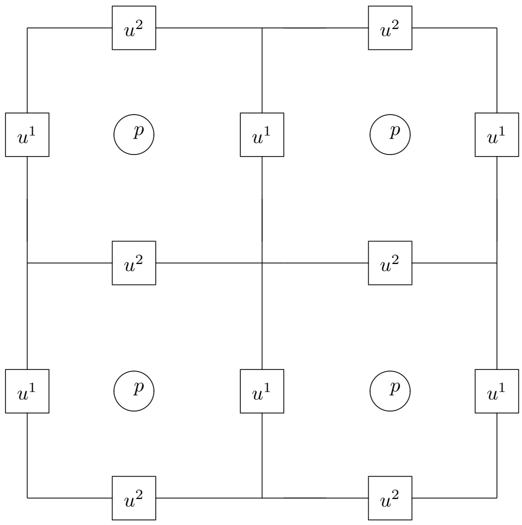

Figure 1. Coarsest staggered grid.

4. Multigrid framework

In the following, we explain the implementation of a proposed full multigrid method to the optimality system (11)–(18). The coarsest staggered grid is shown in Figure 1. In the development of multigrid solver, we face some difficulties due to the coupled state and adjoint systems, and because of the nature of staggered grids.

We note that starting from the given coarse grid, a nested sequence of grids is obtained by tripling the mesh size; see Figure 1. This remark seems novel in the staggered-grid context but it was also used for solving first-order elliptic control problems [**11**] using multigrid methods on staggered grids.

Next, for the multigrid framework, we define a sequence of levels (nested grids) Ω_k of mesh size $h_{xk} = h_{x1}/3^{(k-1)}$, and $h_{y_k} = h_{y_1}/3^{(k-1)}$, $k = 1, \ldots, L$. Here, we denote $k = L$ as the finest level, and $h_{x1} = h_{y_1} = 1/2$ are the mesh sizes of the coarsest grid in the x and y direction, respectively. We denote all operators and functions defined on Ω_k in terms of the index k. By using this setting, a variable X_{IJ}^{k-1} on a grid point (I, J) (of coarse grid Ω_{k-1}) has the same place as the variable X_{ij}^{k} at (i, j) (of the fine grid Ω_k), i.e.,

- $u_{I,J+1/2}^{1,k-1}$ corresponds to $u_{i,j+1/2}^{1,k}$ for $i = 3I - 2$, $j = 3J - 1$,
- $u_{I+1/2,J}^{2,k-1}$ corresponds to $u_{i+1/2,j}^{2,k}$ for $i = 3I - 1$, $j = 3J - 2$,
- $p_{I+1/2,J+1/2}^{k-1}$ corresponds to $p_{i+1/2,j+1/2}^{k}$ for $i = 3I - 1$, $j = 3J - 1$.

4.1. Smoothing scheme

In the following, we present our smoothing scheme that is needed in implementation of the proposed full multigrid algorithm.

To derive a distributive scheme, we note that the state momentum equations are elliptic but the continuity equation is not elliptic [**7**], it is only a part of an elliptic system, i.e., the continuity equation is only a part of elliptic state system. Therefore, the state (adjoint) momentum equations can be relaxed by a classical Gauss-Seidel scheme but for the state (adjoint) continuity equation, we need to relax it by a *distributive relaxation*. In the following, we explain a distributed relaxation technique, applied sequentially to the state and adjoint system with a gradient step to update the control variable.

The so-called reduced cost functional is defined as follows: $\hat{J} := J(\mathbf{u}(\mathbf{f}), \mathbf{f})$, and by $\hat{J}_h$, we mean its discrete counter part in what follows.
Let $(u_h^1, u_h^2, p_h, \lambda_h^1, \lambda_h^2, q_h, f_h^1, f_h^2)$ be the current approximation to the numerical solution. We define and update this approximation by a sequence of iterative steps.

We start with the update of the control functions by performing a gradient update step, i.e.,

$$\begin{aligned} f_h^1 &\leftarrow f_h^1 - t\,\nabla_{f^1} \hat{J}(f_h^1, f_h^2), \\ f_h^2 &\leftarrow f_h^2 - t\,\nabla_{f^2} \hat{J}(f_h^1, f_h^2), \end{aligned}$$

where $\nabla_{f^1} \hat{J}(f_h^1, f_h^2) = \alpha f_h^1 - \lambda_h^1$ and $\nabla_{f^2} \hat{J}(f_h^1, f_h^2) = \alpha f_h^2 - \lambda_h^2$ are the gradients. Moreover, we choose $t = 1$ as a step length during numerical experiments, see Section 5.

Next, we perform the iterative step for the state system. Let (u_h^1, u_h^2, p_h) be the current approximation to the state system (11)–(13). After smoothing (a pointwise Gauss-Seidel relaxation), the residuals of the state momentum equations (11)–(12) by relaxing all the interior points, where u_h^1 and u_h^2 are defined, we now need to smooth the error in the state continuity equation (13) by *distributive relaxation* [**7**]. It is done as follows:

Let $\mathbf{x}$ be the current cell center and

$$r_h^0 = 0 - (\partial_x^h u_h^1 + \partial_y^h u_h^2) \tag{19}$$

be the residual at cell center just before relaxing there. Then the relaxation step is done by the following nine changes

$$\begin{aligned} u_h^j &\leftarrow u_h^j - \delta_p h \partial_j^h \chi_{\mathbf{x}}^h, \\ p_h &\leftarrow p_h + \delta_p h Q_h \chi_{\mathbf{x}}^h, \end{aligned}$$

where $\chi_{\mathbf{x}}^h$ is the characteristic function of the cell center $\mathbf{x}$ and

$$\delta_p = \frac{h}{4} r_h^0. \tag{20}$$

The changes above and δ_p are such that after changing, r_h^0 vanishes. It is easy to see that the pressure changes are such that the state momentum equations residuals

$$r_h^j = f_h^j - Q_h u_h^j - \partial_j^h p_h, \qquad (j = 1, 2), \tag{21}$$

at all points are preserved, regarding Q_h as locally constant. Near the boundary we need to modify δ_p because it is not possible to preserve r_h^j while relaxing the continuity equation (13), see [**7, 8**].

Next, we consider the adjoint system and relax it in the analogous way as done for the state system: Let $(\lambda_h^1, \lambda_h^2, q_h)$ be the current approximation of the numerical solution of adjoint system (14)–(16). The residuals of the adjoint momentum equations (14)–(15) at all the interior points, where λ_h^1 and λ_h^2 are defined, are relax by Gauss-Seidel scheme. The adjoint continuity equation (16), i.e., the variable q_h, is relaxed analogous to the state continuity equation. This completes one relaxation step for the whole optimality system.

4.2. Intergrid transfer operators

We use bilinear interpolation as a prolongation operator. For example, consider the space $\mathcal{U}_k$ of $u^k\colon \Omega_k^{ev} \to \mathbb{R},\ k = 1, \ldots, L$. Between two grids Ω_k and Ω_{k-1}, we define a prolongation operator, $I_{k-1}^k\colon \mathcal{U}_{k-1} \to \mathcal{U}_k$, that is consistent with the assumption of bilinear finite elements on each rectangular partition of the discretization.

Note that in the coarsening by a factor-of-three strategy, the coarse-grid points are the fine-grid points [**9, 10, 11**], see Figure 1. Therefore, to take an advantage of this fact, we use straight injection operator $I_k^{k-1}\colon \mathcal{U}_k \to \mathcal{U}_{k-1}$ for transfer of residuals and solution functions from fine to coarse grids. Here we remark that it is not necessary to use the straight injection operator. We use the straight injection as a restriction operator because it gives a natural choice in a coarsening by a factor-of-three strategy on staggered grids.

Next, we consider the optimality system (11)–(18) at the discretization level k for the unknown variables $x_k = (u_k^1, u_k^2, p_k, \lambda_k^1, \lambda_k^2, q_k, f_k^1, f_k^2)$, i.e.,

$$A_k(x_k) = f_k. \tag{22}$$

A full multigrid (FMG) method is used with full approximation scheme cycle (FAS) as a nested iteration, i.e., the FMG scheme is obtained by combining a nested iteration strategy with the FAS, see [**29**].

Algorithm. FMG for solving $A_L(x_L) = f_L$.

1. *For* $l = K < L$*, set initial approximation* u_l*,*
2. *If* $l < L$*, then interpolate to the next finer working level:* $\tilde{x}_{l+1} = I_l^{l+1} x_l$*,*
3. *Apply FAS to solve* $A_{l+1}(x_{l+1}) = f_{l+1}$*, starting with* $\tilde{x}_{l+1}$*,*
4. *Set* $l := l + 1$*. If* $l < L$*, go to step 2, else stop.*

5. Numerical experiments

In this section, we test our proposed solver on the following two examples to demonstrate the efficiency of the proposed staggered grid multigrid solver. We run our numerical experiments using Matlab (R2017a) on laptop $i7$ with 1.86 GHz and $4GB$ RAM.

5.1. Test 1: Navier-Stokes problem

First, we consider the lid-driven cavity problem

$$\begin{aligned} -\nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega := [0,1]^2, \\ -\nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ u^1 &= 1 && \text{on } \Gamma_1 := [0,1] \times \{1\}, \\ \mathbf{u} &= 0 && \text{on } \Gamma \setminus \Gamma_1. \end{aligned}$$

We apply the proposed FMG Algorithm 4.2 with the smoothing scheme given in Section 4.1. We employ W-cycles with 3-pre and 3-post smoothing steps and stop when $\max\left\{\|r_h^j\|_{L^2}\right\} < 10^{-6}$. In Table 1, maximum number of outer iterations (W-cycles) with CPU time are reported. Furthermore, the velocity field (u^1, u^2) and pressure p with viscosity $\nu = 0.1$ are also depicted in Figure 2. We choose lid-driven cavity problem because it is widely considered for the testing of solvers for Navier-Stokes equations, e.g., see [**14**].

Table 1. Lid-driven cavity problem: Iteration history and CPU time (sec).

$N_x \times N_y$	Itr	CPU
18×18	5	0.07
54×54	6	0.29
162×162	7	2.07

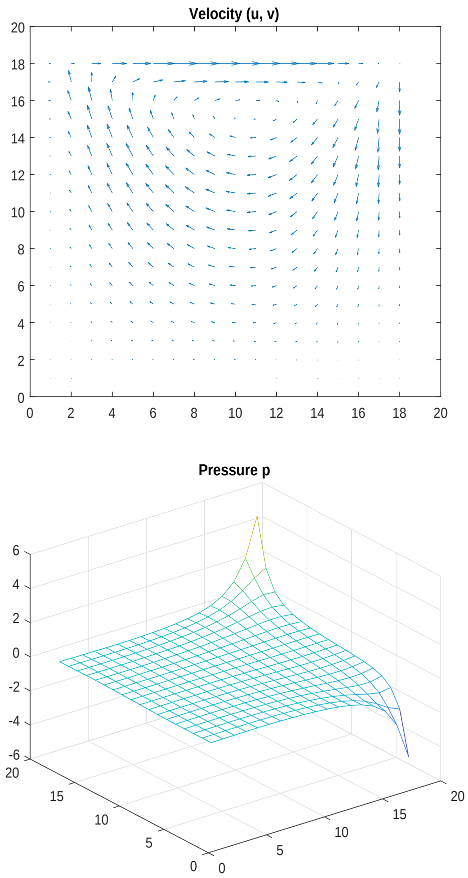

Figure 2. Lid-driven cavity problem: Solution plot of velocity (left) and pressure (right) with viscosity $\nu = 0.1$ on 18×18 mesh.

5.2. Test 2: Navier-Stokes control problem

We consider the distributed Navier-Stokes control problem (1)–(4) with a rectangular domain $\Omega = (0, 1)^2$ and take

$$u_d^1(x, y) := -2x^2y(1-x)^2(1-3y+2y^2),$$
$$u_d^2(x, y) := 2xy^2(1-y)^2(1-3x+2x^2)$$

as the desired state (velocity field).

To solve this control problem, we use our multigrid scheme to the optimality system (11)–(18). We employ W-cycles with 3-pre and 3-post smoothing steps. We use the step length $t = 1$ in the gradient update step and viscosity $\nu = 0.1$. We stop the iterations when the discrete L^2-norm of the residuals satisfies $\max\left\{\|r_h^j\|_{L^2}\right\} < 10^{-6}$.

In Table 2, we report L^2-norm of gradients and number of outer iterations (W-cycles), which are robust in grid size and the regularization parameter. Here we remark that the iteration counts we have observed in this article show better results (robust in grid size h and regularization parameter α) at least for moderate values of α with $\nu = 0.1$, as compared to the recent work [**24**] in which preconditioned iterative methods for Navier-Stokes control problem are presented.

Table 2. L^2-norm of gradients and iterations history.

$N_x \times N_y$	$\|\nabla_{\mathbf{f}} \hat{J}_h\|_{L^2}$	Itr	CPU
$\alpha = 10^{-1}$			
18×18	$2.0557e-07$	5	0.24
54×54	$8.3444e-08$	4	0.64
162×162	$1.1570e-07$	2	1.88
$\alpha = 10^{-2}$			
18×18	$2.2596e-07$	41	1.43
54×54	$2.0844e-07$	24	3.15
162×162	$1.6994e-07$	8	7.34
$\alpha = 10^{-3}$			
18×18	$6.7281e-07$	182	5.29
54×54	$1.8271e-07$	122	14.31
162×162	$1.4218e-07$	16	25.53
$\alpha = 10^{-4}$			
18×18	$6.2549e-07$	239	14.76
54×54	$1.9420e-07$	391	42.77
162×162	$1.0405e-07$	136	129.86
$\alpha = 10^{-5}$			
18×18	$3.5493e-07$	129	45.89
54×54	$1.4282e-07$	126	53.16
162×162	$8.7336e-08$	79	109.15

In Table 3, we report numerical values of tracking errors for different regularization parameter α on $N_x \times N_y = 162 \times 162$. We obtain improved tracking for smaller values of α.

Table 3. L^2-norm of the tracking errors.

α	$\|\mathbf{u} - \mathbf{u_d}\|_{L^2}$
10^{-1}	$4.0455e-03$
10^{-2}	$1.2581e-03$
10^{-3}	$2.2484e-04$
10^{-4}	$4.4967e-05$
10^{-5}	$7.3386e-06$

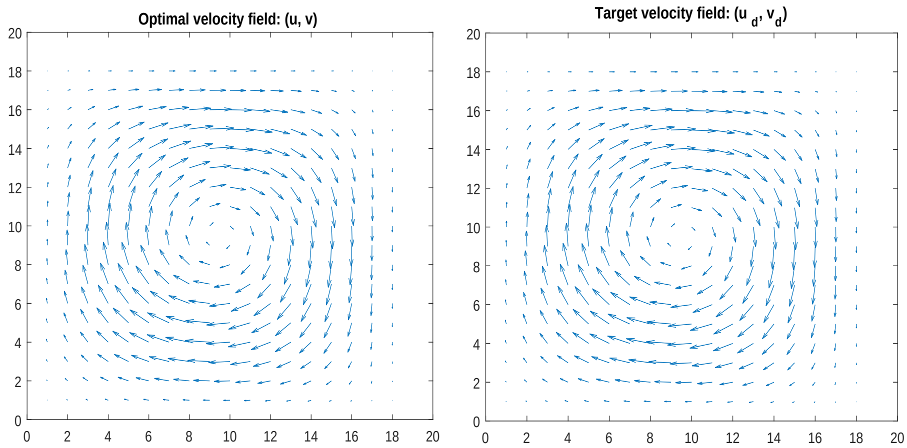

Figure 3. Navier-Stokes control problem: Optimal velocity field (left) and target velocity field (right) with $\alpha = 10^{-5}$, $\nu = 0.1$ on 18×18 mesh.

6. Control-constrained Navier-Stokes control problems

In this section, we extend our proposed multigrid scheme to control-constrained problems, and report results of numerical experiments. For this purpose, consider the (distributed) Navier-Stokes control problem (1)–(4) in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ with control

$$\mathbf{f} \in \mathbf{F}_{ad} = \left\{ \mathbf{u} \in \mathbf{L}^2(\Omega) : \underline{\mathbf{u}} \leq \mathbf{u}(\mathbf{x}) \leq \overline{\mathbf{u}} \text{ a.e. in } \Omega \right\}. \tag{23}$$

The existence and uniqueness of the optimal solution to this problem is standard [**13, 20, 22, 28**] and characterized by the following optimality system

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \quad \text{in } \ \Omega, \\ -\nabla\cdot\mathbf{u} = 0 & \quad \text{in } \ \Omega, \\ \mathbf{u} = 0 & \quad \text{on } \ \Gamma, \end{aligned} \tag{24}$$

$$\begin{aligned} -\nu\,\Delta\lambda - (\mathbf{u}\cdot\nabla)\lambda + (\nabla\mathbf{u})^T\lambda + \nabla q = \mathbf{u_d} - \mathbf{u} & \qquad \text{in } \Omega, \\ \nabla\cdot\lambda = 0 & \qquad \text{in } \Omega, \\ \lambda = 0 & \qquad \text{on } \Gamma, \end{aligned} \tag{25}$$

$$(\alpha\mathbf{f} - \lambda, \tilde{\mathbf{f}} - \mathbf{f}) \geq 0 \qquad \text{for all } \tilde{\mathbf{f}} \in \mathbf{F}_{ad}. \tag{26}$$

The discretization scheme described in Section 3 is also applicable to above optimality system (24)–(26). Hence equations (11)–(16) with the following discretized optimality condition constitute the discretized optimality system for control-constrained problem

$$(\alpha f^1_{i,j+1/2} - \lambda^1_{i,j+1/2}, \tilde{f^1}_{i,j+1/2} - f^1_{i,j+1/2}) \geq 0 \quad \text{in} \quad \Omega^{ev}_h,$$
$$(\alpha f^2_{i+1/2,j} - \lambda^2_{i+1/2,j}, \tilde{f^2}_{i+1/2,j} - f^2_{i+1/2,j}) \geq 0 \quad \text{in} \quad \Omega^{eh}_h,$$

for all $(\tilde{f^1}_{i,j+1/2}, \tilde{f^2}_{i+1/2,j}) \in \mathbf{F}_{adh}$, where $\mathbf{F}_{adh}$ is the discrete analogue of $\mathbf{F}_{ad}$.

For smoothing, first we update the control $\mathbf{f}^h$ by performing a gradient update which includes projection on the control constraint, i.e.,

$$\mathbf{f}^{*,h} := \mathbb{P}_{[\underline{\mathbf{f}},\overline{\mathbf{f}}]}\left(\mathbf{f}^h - t(\alpha\mathbf{f}^h - \lambda^h)\right), \tag{27}$$

where $\nabla\hat{J}(\mathbf{f}^h) = \alpha\mathbf{f}^h - \lambda^h$ and projection $\mathbb{P}_{[a,b]}$ is defined as follows:

$$\mathbb{P}_{[\underline{\mathbf{f}},\overline{\mathbf{f}}]}(\mathbf{f}) = \begin{cases} \overline{\mathbf{f}} & \text{if } \mathbf{f} > \overline{\mathbf{f}}, \\ \mathbf{f} & \text{if } \underline{\mathbf{f}} \leq \mathbf{f} \leq \overline{\mathbf{f}}, \\ \underline{\mathbf{f}} & \text{if } \mathbf{f} < \underline{\mathbf{f}}. \end{cases}$$

Corresponding to the new values of controls, we update the state and adjoint variables using the smoothing scheme as given in Subsection 4.1.

Next, we consider the numerical problem given in previous Section 5 with control-constraints $-0.01 \leq \mathbf{f} \leq 0.01$. To solve this problem, we use our multigrid scheme and employ W-cycles with 3-pre and 3-post smoothing steps with the same stopping criterion as discussed for the unconstrained control problem.

The iteration counts with CPU time (seconds) are reported in Table 4, which demonstrate the efficiency of the proposed FMG method for the control-constrained problems. In Table 5, we report L^2-norm of tracking errors for different values of regularization parameter on $N_x \times N_y = 162 \times 162$ mesh. We obtain slightly better tracking for smaller values of the control weight α. To see the active control-constraints, the control function $\mathbf{f} = (f^1, f^2)$ for $\alpha = 10^{-1}$, is also depicted in Figure 4.

7. Conclusions

We have developed a multigrid algorithm with coarsening by a factor of three to solve distributed optimal control problems constrained by stationary Navier-Stokes equations. The potential advantage of the proposed multigrid solver is the fact that coarsening by a factor of three results in nested hierarchy of staggered grids simplifies the intergrid transfer operators, reduces number of levels, and thus the computations. Results of numerical experiments demonstrate the efficiency of the proposed multigrid solver for distributed control problems with and without control-constraints. The iteration counts we have observed in this article show robust results in grid size h and regularization parameter α, at least for moderate values of regularization parameter α. In future work, one can use second-order upwinding schemes [**8**] and can extend the proposed multigrid scheme for small values of viscosity ν. On the other hand, the proposed multigrid scheme can be

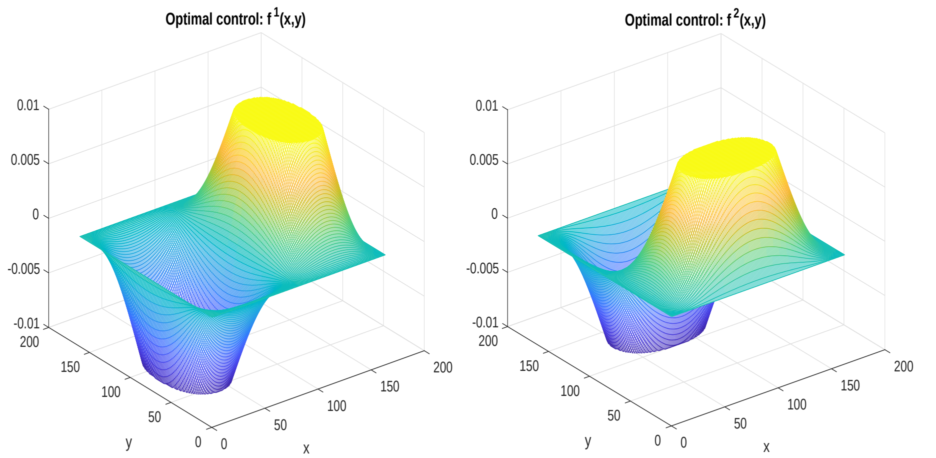

Figure 4. Control-constrained problem: Control function f^1 (left) and f^2 (right) with $\alpha = 10^{-1}$ on 162×162 mesh.

Table 4. Control-constrained problem: Number of iterations and CPU time (sec).

$N_x \times N_y$	$\|\nabla_{\mathbf{f}} \hat{J}_h\|_{L^2}$	Itr	CPU
$\alpha = 10^{-1}$			
18×18	$2.3580e-04$	5	0.22
54×54	$2.3368e-04$	4	0.65
162×162	$2.3342e-04$	3	2.44
$\alpha = 10^{-2}$			
18×18	$7.0586e-04$	16	0.52
54×54	$7.0313e-04$	18	1.99
162×162	$7.0266e-04$	15	9.97
$\alpha = 10^{-3}$			
18×18	$7.7004e-04$	22	0.64
54×54	$7.6679e-04$	33	3.51
162×162	$7.6633e-04$	29	19.11
$\alpha = 10^{-4}$			
18×18	$7.7676e-04$	23	0.68
54×54	$7.7345e-04$	53	4.77
162×162	$7.7299e-04$	32	22.19
$\alpha = 10^{-5}$			
18×18	$7.7743e-04$	22	0.64
54×54	$7.7411e-04$	67	5.97
162×162	$7.7365e-04$	51	33.64

Table 5. Control-constrained problem: Tracking errors.

α	$\|\mathbf{u} - \mathbf{u_d}\|_{L^2}$
10^{-1}	$4.3102e-03$
10^{-2}	$4.0882e-03$
10^{-3}	$4.0810e-03$
10^{-4}	$4.0808e-03$
10^{-5}	$4.0807e-03$

extended to solve time-dependent analogue of the Navier-Stokes optimal control problems [**19**].

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M. M. Butt, Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia,
e-mail: mmunirbutt@gmail.com