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THE PERFORMANCE OF WONG-ZAKAI APPROXIMATIONS FOR THE INVESTIGATION OF STOCHASTIC DIFFERENTIAL EQUATION MODELS WITH NONLINEAR MULTIPLICATIVE NOISE

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Abstract. In this study, Wong-Zakai approximation method has been applied for the analysis of stochastic differential equations appearing in engineering sciences. Wong-Zakai approximation has been used with similar stochastic approximation methods to compare the approximate solutions of the problems and comment on the performance of the method. Models for lake pollution and computer virus spread under antivirus protection have been used with nonlinear stochastic noise as numerical examples to demonstrate the efficiency of Wong-Zakai method.

1. Introduction

Mathematical modeling studies have mostly shifted from systems of ordinary differential equations to various other types of differential equation systems. Fuzzy, fractional and delay differential equations are some of the types which are used in applications for modeling real life events in biology, engineering, social sciences, and etc. [**8, 9, 14**]. Amongst these modeling approaches, random and stochastic systems should be considered separately since they contain random components for modeling the uncertainty of real life dynamics. Stochastic models are considered essential in modeling finance, population dynamics and physics since the uncertainty inherent to the nature of the event can be modeled efficiently through the use of stochastic processes.

Mathematical models consisting of stochastic equations can often be too complex to analyze, especially in situations with nonlinear, multiplicative, and colored noise. Most of the stochastic models in the literature are investigated through approximation techniques for Ito differential equations such as Euler-Maruyama and Milstein methods. Wong-Zakai convergence method for the solution processes of Stratonovich stochastic differential equations enables the analysis of stochastic

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models through the use of deterministic approximation methods such as Adams-Bashforth and Runge-Kutta methods. The deterministic properties that hold for Stratonovich stochastic integration let the use of deterministic techniques for stochastic equations which means an easier investigation for models with stochastic noise. Applications of Wong-Zakai approximation were given for stochastic differential equations with partial differentiation and reflecting boundary conditions [**4, 22**]. In this study, deterministic models for pollution of a system of lakes [**1**] and spread of a computer virus in a network are analyzed [**15**]. Stochastic noise is added to the deterministic models to obtain systems of stochastic differential equations (SDE) which are examined through Wong-Zakai approximation. Euler and Milstein methods are also used for the Ito SDE versions of the models to compare the results of Wong-Zakai method. Wong-Zakai method can be used with various deterministic approximation methods for the analysis of Stratonovich SDEs and for our study we are using predictor-corrector method where Adams-Bashforth method is used as the predictor and Adams-Moulton method is used as the corrector counterparts [**16**]. Stochastic and random investigation of deterministic models to investigate the random dynamics of events which cannot be modeled through the use of deterministic systems were drawing attention in the last years [**8, 10, 11**]. In this study, we extend this approach to engineering problems along with an alternative method which we believe can be applied to many modeling studies in a variety of research areas.

The outline of the paper can be given as follows. In Section 2, we give a brief outline of the methods used for the investigations. In Section 3, the problems are given along with the results and the graphical interpretations. Finally, the concluding remarks are given with a comparative analysis of the results from Wong-Zakai method and other methods used for comparison.

2. Wong-Zakai approximation

The basic idea behind Wong-Zakai approximate solutions of stochastic differential equations (SDE) is based on using the deterministic method for the estimated increments at every time discretization of a solution process. Consider a Stratonovich stochastic differential equation

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t) \circ \mathrm{d}W_t, \tag{1}$$

where $a(t, X_t)$ is the drift term and $b(t, X_t)$ is the diffusion term with $\circ$ denoting the Stratonovich stochastic integral operator. The more popular stochastic differential equation, Ito stochastic differential equation, is given as

$$\mathrm{d}X_t = \underline{a}(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t, \tag{2}$$

with the drift terms of the two equations satisfying $a(t, x) = \underline{a}(t, x) - \frac{1}{2}b(t, x)\frac{\partial b(t,x)}{\partial x}$ between Ito and Stratonovich differential equations [**5**].

Since the problems in this study are systems of stochastic differential equations, a system of stochastic differential equations (1) is examined. For

$$\mathbf{X}_t = [X_t^1 \; X_t^2 \; \dots X_t^n]^T, \quad \mathrm{d}\mathbf{W}_t = [\mathrm{d}W_{1_t} \; \mathrm{d}W_{2_t} \; \dots \mathrm{d}W_{n_t}]^T$$

the stochastic equation (1) takes the system form

$$\mathrm{d}\mathbf{X}_t = \mathbf{a}(t, \mathbf{X}_t)\mathrm{d}t + \mathbf{b}(t, \mathbf{X}_t) \odot \mathrm{d}\mathbf{W}_t \tag{3}$$

where

$$\mathbf{a}(t, \mathbf{X}_t) = [a_1(t, \mathbf{X}_t) \; a_2(t, \mathbf{X}_t) \dots a_n(t, \mathbf{X}_t)]^T,$$
$$\mathbf{b}(t, \mathbf{X}_t) = [b_1(t, \mathbf{X}_t) \; b_2(t, \mathbf{X}_t) \dots b_n(t, \mathbf{X}_t)]^T,$$

and n denotes the number of equations in the system. The expression $\odot$ in (3) denotes the multiplication of the corresponding elements of the vectors $\mathbf{b}(t, \mathbf{X}_t)$ and $\mathrm{d}\mathbf{W}_t$. Hence,

$$\mathbf{b}(t, \mathbf{X}_t) \odot \mathrm{d}\mathbf{W}_t = [b_1(t, \mathbf{X}_t) \circ \mathrm{d}W_{1_t} \; b_2(t, \mathbf{X}_t) \circ \mathrm{d}W_{2_t} \dots b_n(t, \mathbf{X}_t) \circ \mathrm{d}W_{n_t}]^T.$$

Now, Wong-Zakai approximation is given for the stochastic differential equation (3).

The time interval under consideration $[0, T]$ is assigned a discretization of $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = T$. Wong-Zakai method is based on obtaining the numerical approximation $\widehat{\mathbf{X}}_{t_j}$ of the solution of the Stratonovich SDE for each $[t_j, t_{j+1}]$, using the initial approximation $\widehat{\mathbf{X}}_0 = \mathbf{X}_0$ [**18, 19**]. $\widehat{\mathbf{X}}_{t_{j+1}}$ is obtained as

$$\frac{\mathrm{d}\widehat{\mathbf{X}}_t}{\mathrm{d}t} = \mathbf{a}(t, \widehat{\mathbf{X}}_t) + \frac{1}{\Delta_j}\mathbf{b}(t, \widehat{\mathbf{X}}_t)\Delta\mathbf{W}_j \tag{4}$$

within every $[t_j, t_{j+1}]$, $j = 0, 1, \dots, k-1$ [**6**]. Here, $\Delta_j = t_{j+1} - t_j$ and $\Delta\mathbf{W}_j = \mathbf{W}_{t_{j+1}} - \mathbf{W}_{t_j}$, $j = 0, 1, \dots, k-1$.

Stratonovich stochastic integration defines $\int_0^T X_t \circ \mathrm{d}W_t$ as a random variable in the mean square limit

$$\mathrm{l.i.m} \sum_{i=0}^{k-1} \frac{X_{i+1} - X_i}{2}(W_{i+1} - W_i) \tag{5}$$

as the discretization $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = T$ of $[0, T]$ goes infinitesimal ($k \to \infty$) with the Wiener process W_t [**3**]. This enables the application of some rules of calculus needed for numerical integration to the Stratonovich integral such as $\int_0^T f'(W_t) \circ \mathrm{d}W_t = f(W_T) - f(W_0)$ [**3**]. Using this particular advantage of the Stratonovich integral, we combine Wong-Zakai method with a deterministic method, where we use the predictor corrector method [**16**] with Adams Bashforth as the predictor pair

$$\mathbf{X}_{i+1}^* = \mathbf{X}_i + \frac{h}{2}[3f(t_i, \mathbf{X}_i) - f(t_{i-1}, \mathbf{X}_{i-1})] \tag{6}$$

and Adams Moulton method

$$\mathbf{X}_{i+1} = \mathbf{X}_i + \frac{h}{2}[f(t_i, \mathbf{X}_i) + f(t_{i+1}, \mathbf{X}_{i+1}^*)], \qquad i = 2, 3, \dots, m, \tag{7}$$

(m denotes the number of intervals) as the corrector pair, where the function of f is right hand side of equation (4)

$$f(t,\mathbf{X}) = \mathbf{a}(t,\mathbf{X}(t)) + \frac{1}{\Delta_j}\mathbf{b}(t,\mathbf{X}(t))\Delta\mathbf{W}_j.$$

The initial $\mathbf{X}_2$ is calculated with Runge-Kutta II method for $\mathbf{X}_1 = \mathbf{x}_0$, through $\mathbf{k}_1 = f(t_j,\mathbf{X}_1)$, $\mathbf{k}_2 = f(t_j + h, \mathbf{X}_1 + h\mathbf{k}_1)$, and $\mathbf{X}_2 = \mathbf{X}_1 + \frac{h}{2}[\mathbf{k}_1 + \mathbf{k}_2]$, $j = 0,1,\ldots,k-1$, where h is the step size for the deterministic approximation method.

3. Applications

In this section, application of Wong-Zakai method is given for models for lake pollution and computer virus spread under antivirus protection.

3.1. Lake Pollution Model

Pollution of a system of lakes has been analyzed via deterministic differential equation systems using several methods such as modified Differential transformation, Homotopoy perturbation, and Collocation methods, [**12, 21, 13**]. The essential approach of the lake pollution model is based on the idea of monitoring the effects of the pollution in a single lake within a system of lakes which are connected by rivers and channels. The system is given as follows:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{F_{13}}{V_3}x_3(t) - \frac{F_{31}}{V_1}x_1(t) - \frac{F_{21}}{V_1}x_1(t) + p(t),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{F_{21}}{V_1}x_1(t) - \frac{F_{32}}{V_2}x_2(t),\\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= \frac{F_{31}}{V_1}x_1(t) + \frac{F_{32}}{V_2}x_2(t) - \frac{F_{13}}{V_3}x_3(t).\end{aligned} \tag{8}$$

Here, V_i, $i = 1,2,3$, denote the volume of lake i, and x_i, $i = 1,2,3$ denote the amount of pollutant in lake i, respectively. F_{ji} are the flow rates from lake i to lake j for $i,j \in 1,2,3$. Further details about the deterministic analysis of this model can be found in [**1, 12, 21**]. The values of the parameters and initial conditions necessary for the stochastic analysis were given in the referred study as [**1**]:

$$\begin{aligned}&V_1 = 2900\,mi.^3, \quad V_2 = 850\,mi.^3, \quad V_3 = 1180\,mi.^3,\\ &F_{21} = 18\,\frac{mi.^3}{year}, \quad F_{32} = 18\,\frac{mi.^3}{year}, \quad F_{31} = 20\,\frac{mi.^3}{year}, \quad F_{13} = 38\,\frac{mi.^3}{year}.\end{aligned} \tag{9}$$

The time variable t denotes the number of years. Initial conditions are given as $x_1(0) = 0$, $x_2(0) = 0$, and $x_3(0) = 0$. Implementing the values of F_{ji} and adding nonlinear independent multiplicative stochastic noise give the stochastic model of pollution for a system of lakes:

$$
\begin{aligned}
\mathrm{d}x_1(t) &= \left(\frac{38}{1180}x_3(t) - \frac{38}{2900}x_1(t) + p(t)\right)\mathrm{d}t + (a_1 + b_1x_1)x_1\mathrm{d}W_{1_t},\\
\mathrm{d}x_2(t) &= \left(\frac{18}{2900}x_1(t) - \frac{18}{850}x_2(t)\right)\mathrm{d}t + (a_2 + b_2x_2)x_2\mathrm{d}W_{2_t},\\
\mathrm{d}x_3(t) &= \left(\frac{20}{2900}x_1(t) + \frac{18}{850}x_2(t) - \frac{38}{1180}x_3(t)\right)\mathrm{d}t + (a_3 + b_3x_3)x_3\mathrm{d}W_{3_t}.
\end{aligned} \tag{10}
$$

The role of the stochastic noise $(a_i + b_ix_i)x_i\mathrm{d}W_{i_t}$, $i = 1, 2, 3$, in this model is to demonstrate the total effect of external factors on the spread of pollution in the lake system. The function $p(t)$ denotes the pollutant introduced to the system and for the numerical simulations, we use a periodic pollutant function with $p(t) = 1+\sin t$. The numerical solutions of the stochastic system (10) obtained with Euler-Maruyama, Milstein, stochastic Runge-Kutta, and Wong-Zakai methods are given in the table below (Table 1). The solutions are also shown in the figure with additional solution curves for stochastic strong order 1.0 Runge-Kutta method (Figure 1). Note that the results have been given for $a_i = 0.55$, $b_i = 10^{-4}$, $i = 1, 2, 3$, to simulate a stochastic model with small nonlinear diffusion.

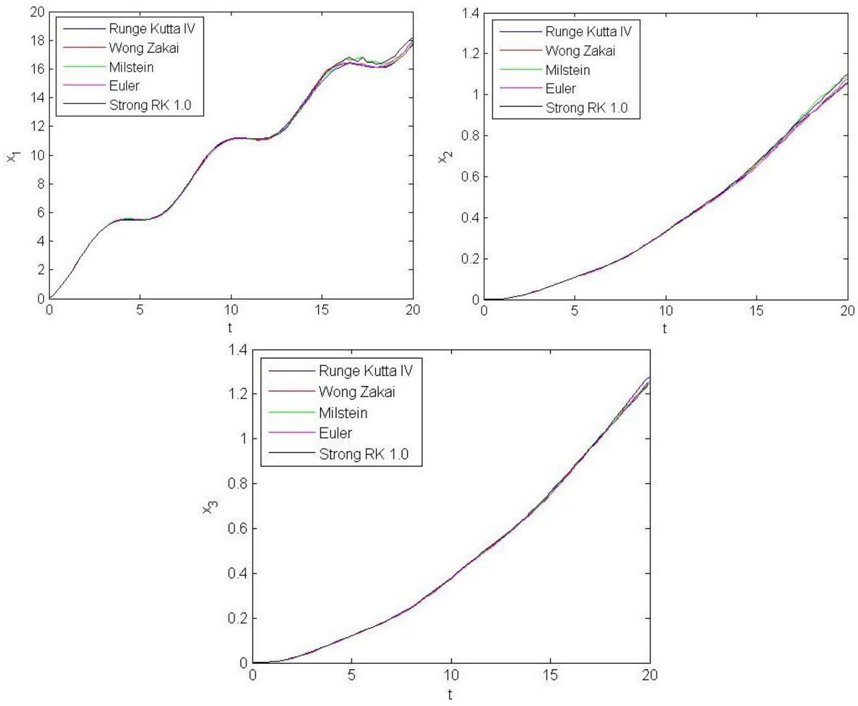

Figure 1. The solutions for the stochastic model (10).

Table 1. Numerical results for the stochastic model (10).

	Runge Kutta IV			Euler- Maruyama		
Time	$E(x_1(t))$	$E(x_2(t))$	$E(x_3(t))$	$E(x_1(t))$	$E(x_2(t))$	$E(x_3(t))$
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0	1.4479	0.0040	0.0045	1.4304	0.0038	0.0042
2.0	3.3696	0.0187	0.0209	3.3538	0.0181	0.0202
3.0	4.8989	0.0440	0.0493	4.8996	0.0432	0.0483
4.0	5.4717	0.0753	0.0845	5.5126	0.0747	0.0841
5.0	5.4773	0.1072	0.1211	5.5039	0.1070	0.1206
6.0	5.7337	0.1392	0.1579	5.7213	0.1386	0.1568
7.0	6.8839	0.1738	0.1992	6.8294	0.1736	0.1967
8.0	8.6167	0.2179	0.2489	8.6419	0.2163	0.2456
9.0	10.2826	0.2720	0.3104	10.2453	0.2705	0.3065
10.0	11.1065	0.3333	0.3804	11.0651	0.3305	0.3762
11.0	11.1564	0.3962	0.4502	11.1266	0.3909	0.4465
12.0	11.1088	0.4558	0.5184	11.1477	0.4493	0.5144
13.0	11.8417	0.5154	0.5883	11.9404	0.5114	0.5867
14.0	13.4921	0.5824	0.6672	13.5446	0.5750	0.6662
15.0	15.1302	0.6620	0.7586	15.3887	0.6497	0.7495
16.0	16.1334	0.7482	0.8549	16.3497	0.7358	0.8527
17.0	16.2700	0.8380	0.9557	16.3459	0.8279	0.9552
18.0	16.1363	0.9124	1.0524	16.1386	0.9094	1.0547
19.0	16.4841	0.9894	1.1531	16.6858	0.9912	1.1570
20.0	17.8059	1.0634	1.2586	17.9781	1.0797	1.2572

	Milstein			Wong-Zakai		
Time	$E(x_1(t))$	$E(x_2(t))$	$E(x_3(t))$	$E(x_1(t))$	$E(x_2(t))$	$E(x_3(t))$
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0	1.4311	0.0038	0.0042	1.4483	0.0040	0.0045
2.0	3.3530	0.0181	0.0202	3.3655	0.0187	0.0209
3.0	4.8917	0.0432	0.0484	4.8865	0.0440	0.0491
4.0	5.5352	0.0746	0.0840	5.4935	0.0755	0.0844
5.0	5.5028	0.1073	0.1209	5.4994	0.1078	0.1206
6.0	5.7037	0.1388	0.1569	5.7609	0.1397	0.1572
7.0	6.7959	0.1740	0.1977	6.8939	0.1748	0.1977
8.0	8.6002	0.2173	0.2483	8.7018	0.2189	0.2478
9.0	10.2788	0.2717	0.3104	10.3900	0.2729	0.3095
10.0	11.0895	0.3320	0.3797	11.0988	0.3329	0.3795
11.0	11.1472	0.3747	0.4507	11.1339	0.3948	0.4492
12.0	11.1697	0.4566	0.5205	11.0729	0.4509	0.5168
13.0	11.8907	0.5185	0.5923	11.9157	0.5096	0.5857
14.0	13.5265	0.5862	0.6688	13.6615	0.5773	0.6624
15.0	15.3479	0.6647	0.7580	15.4388	0.6600	0.7531
16.0	16.3659	0.7556	0.8569	16.3114	0.7365	0.8496
17.0	16.7537	0.8400	0.9598	16.3649	0.8228	0.9590
18.0	16.4914	0.9400	1.0591	16.1335	0.9093	1.0559
19.0	16.5214	1.0153	1.1605	16.4422	0.9815	1.1549
20.0	17.9631	1.0895	1.2498	17.7038	1.0580	1.2495

It is known that the stochastic Runge-Kutta method is of the highest order between these methods, hence we use the results from Runge-Kutta scheme to investigate the results. The solution curves and numerical results suggest that all of these methods provide similar results to one another, meaning Wong-Zakai method performs similarly according to the more popular stochastic methods.

Using the results of Runge-Kutta method as a basis, the relative errors at several time points are found as follows for Wong-Zakai, Euler-Maruyama, and Milstein methods (Table 2). Note that the calculations have been done for $\Delta_j = 0.05$ with $N = 10^5$ simulations for each model.

Table 2. Error percentages relative to stochastic Runge-Kutta results for (10).

	Euler- Maruyama			Milstein			Wong-Zakai		
Time	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$
$t = 5$	0.4856	0.1866	0.4129	0.4656	0.0933	0.1652	0.4035	0.5597	0.4129
$t = 10$	0.3728	0.8401	1.1041	0.1531	0.3900	0.1840	0.0693	0.1200	0.2366
$t = 15$	1.7085	1.8580	1.1996	1.4382	0.4079	0.0791	2.0396	0.3021	0.7250
$t = 20$	0.9671	1.5328	0.1112	0.8829	2.4544	0.6992	0.5734	0.5078	0.7230

It is seen that between these methods, the relative errors (relative to the results from stochastic Runge-Kutta method) are of similar percentages. Euler-Maruyama and Milstein schemes are widely used in applications, whereas it is seen that Wong-Zakai method could provide a competent alternative.

3.2. Computer Virus Model

The spread of malicious software, computer viruses, and rumor in social media and networks were a concentration point for the use of compartmental models lately [**7, 17, 20**]. The similarities between the spread of diseases in populations and malware in networks provide a basis for the use of epidemiological models in analyzing the spread of computer viruses. In this example, the equation system in Onwubuoya et al. [**15, 2**] is used for modeling the stochasticity of the effects of antivirus on computer virus infections. The deterministic model is given as:

$$
\begin{aligned}
\frac{\mathrm{d}X(t)}{\mathrm{d}t} &= aZ(t) - bX(t), \\
\frac{\mathrm{d}Y(t)}{\mathrm{d}t} &= c - \mathrm{d}Y(t) - eX(t)Y(t), \\
\frac{\mathrm{d}Z(t)}{\mathrm{d}t} &= eX(t)Y(t) - (d + f)Z(t) - gN(t), \\
\frac{\mathrm{d}N(t)}{\mathrm{d}t} &= h - iN(t).
\end{aligned} \tag{11}
$$

Here, $X(t)$ denotes the number of worms, $Y(t)$ denotes the number of uninfected files, $Z(t)$ denotes the number of infected files, and $N(t)$ denotes the number of antivirus agents. The initial values of the deterministic model are given as follows: $X(0) = 15$, $Y(0) = 3$, $Z(0) = 20$, and $N(0) = 0.5$ [**15**]. Using these initial values and the deterministic system (11), the stochastic model is given as:

$$
\begin{aligned}
\mathrm{d}X(t) &= (aZ(t) - bX(t))\,\mathrm{d}t + (a_4 + b_4X(t))X(t)\mathrm{d}W_{4_t},\\
\mathrm{d}Y(t) &= (c - \mathrm{d}Y(t) - eX(t)Y(t))\,\mathrm{d}t + (a_5 + b_5Y(t))Y(t)\mathrm{d}W_{5_t},\\
\mathrm{d}Z(t) &= (eX(t)Y(t) - (d+f)Z(t) - gN(t))\,\mathrm{d}t + (a_6 + b_6Z(t))Z(t)\mathrm{d}W_{6_t},\\
\mathrm{d}N(t) &= (h - iN(t))\,\mathrm{d}t + (a_7 + b_7N(t))N(t)\mathrm{d}W_{7_t}.
\end{aligned}
\tag{12}
$$

Once again, the nonlinear diffusion coefficient models the stochastic nature of the antivirus effect. The values and the descriptions of the parameters are given in the referred study as [**15**]:

Table 3. Descriptions and values of parameters.

Parameter	Description	Value
a	Rate of infected files becoming worms	0.3
b	Worm death rate	0.5
c	Uninfected file birth rate	2.3
d	Uninfected file natural death rate	0.055
e	Uninfected file worm-related infection rate	0.015
f	Infected file death rate	0.055
g	Antivirus efficiency rate	0.002
h	Antivirus activity rate	2.6
i	Antivirus inefficiency rate	0.1

The time variable t denotes the number of minutes. The numerical solutions of the stochastic system (12) obtained with Euler-Maruyama, Milstein, stochastic Runge-Kutta, and Wong-Zakai methods are given in the table below (Table 4). The solutions are also shown in the figure with additional solution curves for stochastic strong order 1.0 Runge-Kutta method (Figure 2).

Note that the results are given for $\Delta_j = 0.05$ step size with $N = 10^5$ simulations for each of the model variables. The realizations are averaged to obtain the solutions for expected values of the variables shown above (Figure 2)

$$
E(X_t) = \frac{1}{N}\sum_{i=1}^{N} X_t^i. \tag{13}
$$

Table 4. Results for the expected values of the variables for (12).

	Runge-Kutta IV				Euler			
Time	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$
0.0	15.0000	3.0000	20.0000	0.5000	15.0000	3.0000	20.0000	0.5000
1.0	13.6775	4.3148	18.6619	2.9237	13.5991	4.3079	18.6051	2.9316
2.0	12.5603	5.4101	17.6523	5.1151	12.4547	5.4108	17.5385	5.1434
3.0	11.6689	6.3709	16.7774	7.1115	11.5884	6.3706	16.6723	7.1334
4.0	10.8932	7.2307	16.0017	8.8875	10.8658	7.2457	15.9060	8.9382
5.0	10.2635	8.0630	15.3457	10.5141	10.2271	8.0351	15.2366	10.5663
6.0	9.7608	8.7990	14.8221	11.9682	9.7254	8.7826	14.6028	12.0468
7.0	9.3391	9.5462	14.2881	13.2791	9.2657	9.5166	14.0569	13.4099
8.0	8.9544	10.2400	13.7895	14.5118	8.9123	10.2295	13.5638	14.5990
9.0	8.6459	10.9025	13.3527	15.5746	8.5689	10.8972	13.1250	15.6464
10.0	8.4885	11.5227	12.9862	16.5710	8.2589	11.5118	12.7161	16.6506
11.0	8.3390	12.1259	12.6353	17.4909	7.9608	12.1300	12.3799	17.5056
12.0	8.0844	12.6723	12.2957	18.3171	7.6912	12.7317	12.0563	18.2460
13.0	7.8068	13.2059	11.9351	19.0641	7.5031	13.2820	11.7935	19.0496
14.0	7.3821	13.7144	11.6658	19.7717	7.2628	13.7619	11.4545	19.6434
15.0	7.1847	14.1886	11.4072	20.4210	7.0607	14.3184	11.1487	20.2479
16.0	6.9461	14.6476	11.1211	21.0409	6.8826	14.7552	10.9685	20.8318
17.0	6.7925	15.1144	10.8418	21.5291	6.7513	15.2747	10.7932	21.2934
18.0	6.6574	15.5833	10.5573	21.8791	6.6382	15.7383	10.5700	21.7831
19.0	6.5322	16.0257	10.3463	22.2008	6.4971	16.0917	10.4594	22.1420
20.0	6.3681	16.4307	10.1406	22.4631	6.4154	16.4561	10.2702	22.4813

	Milstein				Wong-Zakai			
Time	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$
0.0	15.0000	3.0000	20.0000	0.5000	15.0000	3.0000	20.0000	0.5000
1.0	13.6258	4.3214	18.6368	2.9267	13.6378	4.3161	18.6757	2.9220
2.0	12.5104	5.4178	17.5815	5.1334	12.5453	5.4122	17.6281	5.1209
3.0	11.6402	6.3809	16.6849	7.1351	11.6893	6.3653	16.7629	7.1003
4.0	10.8673	7.2427	15.9559	8.9179	10.9499	7.2257	15.9743	8.9021
5.0	10.2264	8.0333	15.2902	10.5705	10.3518	8.0249	15.4028	10.5160
6.0	9.7424	8.7695	14.6740	12.0256	9.8332	8.8001	14.8391	11.9986
7.0	9.2642	9.4806	14.1093	13.3517	9.4150	9.4811	14.3273	13.3364
8.0	8.8852	10.1965	13.6029	14.5519	9.0078	10.1862	13.8982	14.5307
9.0	8.5449	10.8877	13.1989	15.6114	8.6782	10.8706	13.3927	15.5910
10.0	8.2684	11.4998	12.8103	16.5641	8.3632	11.5167	13.0144	16.6041
11.0	7.9664	12.0961	12.4352	17.4902	8.0843	12.0836	12.5209	17.5070
12.0	7.7397	12.6556	12.0201	18.3020	7.7737	12.6545	12.1496	18.3002
13.0	7.5599	13.2171	11.7403	19.0070	7.5372	13.2563	11.8557	18.9597
14.0	7.3225	13.7922	11.4334	19.6322	7.3100	13.7638	11.5972	19.6228
15.0	7.0794	14.2995	11.2056	20.1765	7.1122	14.2978	11.3466	20.2024
16.0	6.9099	14.7981	10.9012	20.7185	6.9746	14.7517	11.0147	20.8144
17.0	6.6985	15.2255	10.6340	21.0963	6.8279	15.2110	10.7523	21.2351
18.0	6.5195	15.6081	10.4319	21.5907	6.6099	15.5995	10.5412	21.7097
19.0	6.3834	16.0799	10.2445	21.9924	6.4278	15.9882	10.3593	22.1495
20.0	6.2967	16.4414	10.0856	22.3149	6.3249	16.3947	10.1975	22.5071

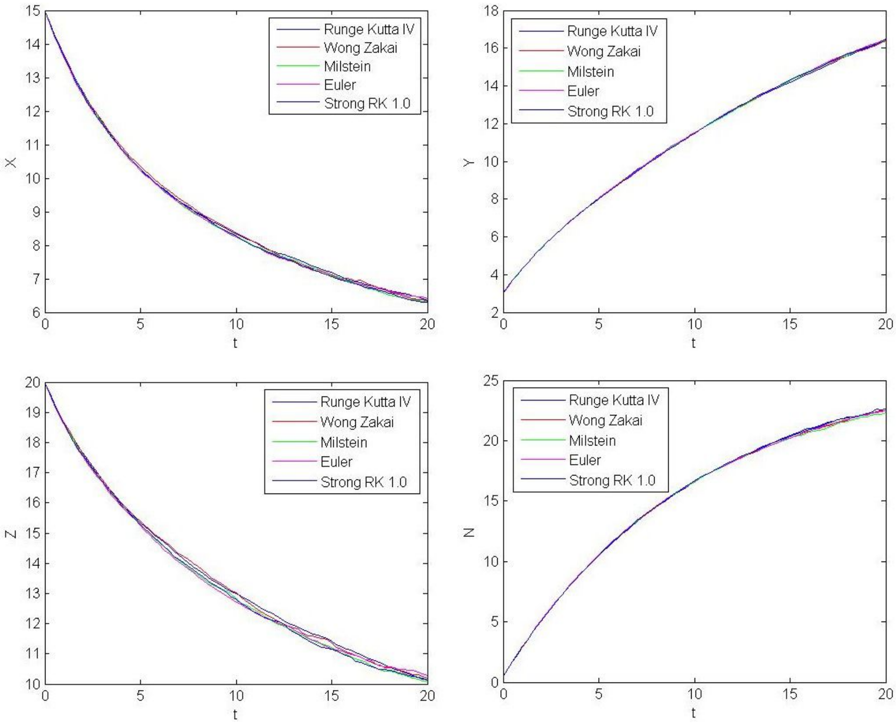

Figure 2. The solutions for the stochastic model (12).

The relative errors (relative to stochastic Runge-Kutta method) are obtained as follows (Table 5):

Table 5. Error percentages relative to stochastic Runge-Kutta results for (12).

	Euler				Milstein			
Time	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$
5.0	0.3547	0.3460	0.7109	0.4965	0.3615	0.3683	0.3617	0.5364
10.0	0.9605	0.0946	2.0799	0.4804	0.8466	0.1987	1.3545	0.0416
15.0	1.7259	0.9148	2.2661	0.8477	1.4656	0.7816	1.7673	1.1973
20.0	0.7428	0.1546	1.2780	0.0810	1.1212	0.0651	0.5424	0.6597

	Wong-Zakai			
Time	$X(t)$	$Y(t)$	$Z(t)$	$N(t)$
5.0	0.8603	0.4725	0.3721	0.0181
10.0	0.2902	0.0521	0.2172	0.1997
15.0	1.0091	0.7696	0.5312	1.0705
20.0	0.6784	0.2191	0.5611	0.1959

It is seen that Wong-Zakai approximation provides the best convergence relative to the higher ordered Runge-Kutta scheme at certain points. The figures also suggest that all of the methods provide similar results for the variables of the stochastic model. Note that the results have been given for $a_i = 0.45$,$b_i = 10^{-4}, i = 4, 5, 6, 7$ to simulate a stochastic model with small nonlinear diffusion.

4. Conclusion

In this study, the performance of Wong-Zakai approximation method has been investigated in comparison with more popular stochastic approximation methods such as Euler-Maruyama, Milstein, and stochastic Runge-Kutta methods. Two numerical examples have been analyzed to compare the results from these methods using solution graphs and relative errors of the results. Additional solution curves have been added for the strong order 1.0 stochastic Runge-Kutta scheme to underline the similarity between the results from the more popular stochastic methods and Wong-Zakai method. Wong-Zakai convergence can be applied to obtain the approximate solution processes of Stratonovich stochastic differential equations. Using the deterministic properties of Stratonovich stochastic integration, deterministic approximation methods are used jointly with stochastic Wong-Zakai approximation to obtain the approximate solution process. Nonlinear stochastic noise has been added to both deterministic models, the Lake Pollution Model, and the Computer Virus Model to obtain stochastic models for numerical examples. The Stratonovich stochastic differential equation versions of the stochastic models have been analyzed with Wong-Zakai approximation using the deterministic Predictor-Corrector method as the deterministic part of the scheme. Adams-Bashforth method has been used as the predictor and Adams-Moulton method has been used as the corrector counterpart of the method. The stochastic expected values of the solutions have been obtained by averaging $N = 10^5$ realizations of the stochastic approximate solutions for both examples.

Numerical results, relative errors, and solution graphs show that Wong-Zakai method performs just as well as the other stochastic methods. Relative errors, relative to stochastic Runge Kutta IV method, show that Wong-Zakai method provide the best results for some of the variables in the stochastic models at certain moments. Results for $t = 5, 10, 15$, and 20, have been given to show the similarity between the absolute error percentages for the methods. For instance, Wong-Zakai produces the best result relative to Runge-Kutta method for the approximate expected value of $N(t)$ at $t = 5.0$ with 0.0181% error, whereas Euler-Maruyama and Milstein methods give 0.4965%, and 0.5364% errors, respectively. There are moments where Euler and Milstein methods provide results with smaller error percentages but Figures 1 & 2 show the similarity between the results of all models. It should be noted that these results were given for $a_i = 0.55$ in the first example, $a_i = 0.45$ in the second example and $b_i = 10^{-4}$ for the nonlinear noise coefficient. These values have been chosen to simulate stochastic models under the influence of nonlinear stochastic noise with small diffusion coefficients. The nonlinear stochastic noise in the equation systems may cause convergence issues for the approximation methods but it is seen that Wong-Zakai performs just as good as the other stochastic methods. In addition, due to the deterministic properties of Stratonovich integration, the performance of Wong-Zakai method becomes increasingly better for a smaller diffusion coefficient, i.e., the method performs better as the problem becomes more deterministic. It is known that most of the stochastic models and problems in the literature are analyzed by using Euler-Maruyama and Milstein methods. However, the results show that Wong-Zakai method performs similarly compared to these models and can be used as an alternative approximation technique.

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