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TOTAL VERTEX-EDGE DOMINATION IN TREES

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Abstract. A subset $S \subseteq V$ is a dominating set of G if every vertex in $V \setminus S$ has a neighbor in S, and it is a total dominating set if every vertex in V has a neighbor in S. The total domination number of G, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A vertex v of a graph G is said to ve-dominate every edge incident to v, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a vertex-edge dominating set (or simply, a ve-dominating set) if every edge of E is ve-dominated by at least one vertex of S. A total ve-dominating set of G is a ve-dominating set whose induced subgraph has no isolated vertex. The vertex-edge domination number $\gamma_{ve}(G)$ is the minimum cardinality of a total ve-dominating set and the total vertex-edge domination number $\gamma^t_{ve}(G)$ is the minimum cardinality of a total ve-dominating set in G. In this paper, we characterize all trees T with $\gamma^t_{ve}(T) = \gamma_t(T)$ or $\gamma^t_{ve}(T) = \gamma_{ve}(T)$, answering two open problems posed in [Boutrig R. and Chellali M., *Total vertex-edge domination*, Int. J. Comput. Math. **95** (2018), 1820–1828]. Moreover, we show that it is NP-hard to decide whether $\gamma^t_{ve}(G) = \gamma_{ve}(G)$ for a given connected $(K_4 - e)$-free graph G.

1. Introduction

In this paper, G is a simple nontrivial connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V : uv \in E\}$, the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$, and the *degree* of v is $\deg_G(v) = |N(v)|$. A vertex of degree one is called a *pendant vertex* or a *leaf* and its neighbor is called a *support vertex.* A *strong support vertex* is a support vertex adjacent to at least two leaves and an *end support vertex* is a support vertex having at most one non-leaf neighbor. A *pendant path* P in G is an induced path such that one of the endpoints has degree one in G, and its other endpoint is the only vertex of P adjacent to some vertex in $G - P$. The *distance* between two vertices u and v in a connected graph G is the length of a shortest uv-path in G.

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The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively, and let $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We write P_n for the *path* of order n. A *double star* $DS_{p,q}$ is a tree obtained from $K_{1,p}$ and $K_{1,q}$ by connecting the center of $K_{1,p}$ with that of $K_{1,q}$. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$ and the sum $f(V(G))$ is called the *weight* $\omega(f)$ of f.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to a vertex in S, and it is a *total dominating set* **(TDS)** if every vertex in V is adjacent to a vertex in S. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. A total dominating set of G with minimum cardinality is called a $\gamma_t(G)$*-set.* Total domination was introduced by Cockayne, Daws, and Hedetniemi [**6**]. The reader is referred to Henning and Yeo's book [**7**] for more details on total domination. In additional, we refer to [**1, 2, 3**].

A vertex v *ve-dominates* every edge incident to any vertex in $N[v]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply, a *ve-dominating set*), if for every edge $e \in E$, there exists a vertex $v \in S$ that *ve*-dominates e. The *vertex-edge domination number,* $\gamma_{ve}(G)$ of G, is the minimum cardinality of a *ve*-dominating set of G. Vertex-edge domination was introduced by Peters [**12**] in his 1986 PhD thesis, and studied further in [**5, 8, 9, 10, 11**]. It is worth noting that Lewis showed that the decision problem corresponding to the problem of computing $\gamma_{ve}(G)$ is NP-complete for bipartite graphs, and it is linearly solvable for trees.

A total *ve*-dominating set (or simply, *total ve-dominating set*) of G is a *ve*-dominating set whose induced subgraph has no isolated vertex. The *total vertex-edge domination number* $\gamma^t_{ve}(G)$ is the minimum cardinality of a total *ve*-dominating set of G. The concept of total vertex-edge domination in graphs was introduced by Boutrig and Chellali in [**4**], who showed that the decision problem corresponding to the problem of computing $\gamma^t_{ve}(G)$ is NP-complete for bipartite graphs. They also observed that for any nontrivial connected graph G,

$$\gamma_{ve}(G) \le \gamma^t_{ve}(G) \le \gamma_t(G). \tag{1}$$

Moreover, they posed the following open problems.

Problem 1. Characterize the nontrivial connected graphs G with $\gamma^t_{ve}(G) = \gamma_t(G)$.

Problem 2. Characterize the nontrivial connected graphs G with $\gamma^t_{ve}(G) = \gamma_{ve}(G)$.

In this paper, we settle the above problems for trees by providing a characterization of all trees T with $\gamma^t_{ve}(T) = \gamma_t(T)$ or $\gamma^t_{ve}(T) = \gamma_{ve}(T)$. Moreover, we show that it is NP-hard to decide whether $\gamma^t_{ve}(G) = \gamma_{ve}(G)$ for a given connected $(K_4 - e)$-free graph G.

2. Preliminaries

In this section, we provide some definitions and observations that are useful throughout the paper.

Definition 2.1. Let u be a vertex of a graph G. A subset S of vertices is said to be an *almost total ve-dominating set* with respect to u if the conditions: (i) any edge not incident to u is ve-dominated by a vertex in S, (ii) every vertex in $S \setminus \{u\}$ is adjacent to a vertex in S, are fulfilled. Define

$$\gamma_{ve}^t(G; u) = \min\{|S|. \ S \text{ is an almost total } ve\text{-dominating set with respect to } u\}.$$

Clearly, any total ve-dominating set in G is an almost total ve-dominating set with respect to any vertex of G. Hence $\gamma_{ve}^t(G; u)$ is well defined, and thus $\gamma_{ve}^t(G; u) \leq \gamma_{ve}^t(G)$ for each $u \in V(G)$. Define

$$W_G^1 = \{u \in V \mid \gamma_{ve}^t(G; u) = \gamma_{ve}^t(G)\}.$$

Definition 2.2. For a graph G, define

$$W_G^2 = \{v \in V \mid v \text{ belongs to no } \gamma_{ve}(G)\text{-set}\}.$$

Definition 2.3. Let u be a vertex of a graph G. A subset $S \subseteq V$ is said to be an *almost ve-dominating set* with respect to u, if any edge not incident to u is ve-dominated by a vertex in S. Suppose

$$\gamma_{ve}(G; u) = \min\{|S| \ ; \ S \text{ is an almost } ve\text{-dominating set with respect to } u\}.$$

Since any ve-dominating set on G is an almost ve-dominating set with respect to any vertex of G, $\gamma_{ve}(G; u)$ is well defined and thus $\gamma_{ve}(G; u) \leq \gamma_{ve}(G)$ for each $u \in V$. Let

$$W_G^3 = \{u \in V \mid \gamma_{ve}(G; u) = \gamma_{ve}(G)\}.$$

Definition 2.4. For a graph G, define

$$W_G^4 = \{v \in V \mid v \text{ belongs to some } \gamma_{ve}^t(G)\text{-set}\}.$$

Proposition 2.5. *Let G be a nontrivial connected graph and let u be a vertex of G. If G' is the graph obtained from G by adding a path $P_4 = x_1x_2x_3x_4$ and joining u to x_1, then $\gamma_t(G') = \gamma_t(G) + 2$ and $\gamma_{ve}^t(G') \leq \gamma_{ve}^t(G) + 2$.*

Proof. Clearly, any $\gamma_t(G)$-set can be extended to a TDS of G' by adding x_2, x_3, and so $\gamma_t(G') \leq \gamma_t(G) + 2$. To prove the inverse inequality, let $v \in N_G(u)$ and D be a $\gamma_t(G')$-set containing no leaves. Then $x_2, x_3 \in D$ and $D \cap N_G[v] \neq \emptyset$. If $x_1 \notin D$, then $D \setminus \{x_2, x_3\}$ is a TDS of G, yielding $\gamma_t(G') \geq \gamma_t(G) + 2$. Hence assume that $x_1 \in D$. Then $(D \setminus \{x_1, x_2, x_3\}) \cup \{u\}$ when $v \in D$, or $(D \setminus \{x_1, x_2, x_3\}) \cup \{v\}$ when $v \notin D$, is a TDS of G, implying that $\gamma_t(G') \geq \gamma_t(G) + 2$. Hence $\gamma_t(G') = \gamma_t(G) + 2$.

Also, any $\gamma_{ve}^t(G)$-set can be extended to a total ve-dominating set of G' by adding x_2, x_3, and this implies that $\gamma_{ve}^t(G') \leq \gamma_{ve}^t(G) + 2$. □

Observation 2.6. *Let H be a subgraph of a graph G. If $\gamma_{ve}^t(H) = \gamma_t(H)$, $\gamma_t(G) \le \gamma_t(H) + s$, and $\gamma_{ve}^t(G) \ge \gamma_{ve}^t(H) + s$ for some non-negative integer s, then $\gamma_{ve}^t(G) = \gamma_t(G)$.*

Proof. We deduce from the assumptions that

$$\gamma_{ve}^t(G) \ge \gamma_{ve}^t(H) + s \ge \gamma_{ve}^t(H) + s \ge \gamma_t(G)$$

which leads to the desired result by (1). □

Observation 2.7. *Let H be a subgraph of a graph G. If $\gamma_{ve}^t(G) = \gamma_t(G)$, $\gamma_{ve}^t(G) \le \gamma_{ve}^t(H) + s$, and $\gamma_t(G) \ge \gamma_t(H) + s$ for some non-negative integer s, then $\gamma_{ve}^t(H) = \gamma_t(H)$.*

Proof. By inequality (1) and the assumptions, we have

$$\gamma_t(G) = \gamma_{ve}^t(G) \le \gamma_{ve}^t(H) + s \le \gamma_t(H) + s \le \gamma_t(G).$$

Thus all inequalities in the above chain must be equalities. In particular, $\gamma_{ve}^t(H) = \gamma_t(H)$. □

Observation 2.8. *Let H be a subgraph of a graph G. If $\gamma_{ve}^t(H) = \gamma_{ve}(H)$, $\gamma_{ve}^t(G) \le \gamma_{ve}^t(H) + s$, and $\gamma_{ve}(G) \ge \gamma_{ve}(H) + s$ for some non-negative integer s, then $\gamma_{ve}^t(G) = \gamma_{ve}(G)$.*

Proof. We deduce from the assumptions that

$$\gamma_{ve}(G) \ge \gamma_{ve}(H) + s = \gamma_{ve}^t(H) + s \ge \gamma_{ve}^t(G)$$

which leads to the result by (1). □

Observation 2.9. *Let H be a subgraph of a graph G. If $\gamma_{ve}^t(G) = \gamma_{ve}(G)$, $\gamma_{ve}(G) \le \gamma_{ve}(H) + s$, and $\gamma_{ve}^t(G) \ge \gamma_{ve}^t(H) + s$ for some non-negative integer s, then $\gamma_{ve}^t(H) = \gamma_{ve}(H)$.*

Proof. By the assumptions and inequality (1), we have

$$\gamma_{ve}^t(G) = \gamma_{ve}(G) \le \gamma_{ve}(H) + s \le \gamma_{ve}^t(H) + s \le \gamma_{ve}^t(G).$$

Thus all inequalities occurring in the above chain, must be equalities. In particular, $\gamma_{ve}^t(H) = \gamma_{ve}(H)$. □

We close this section with two simple observations.

Observation 2.10. *If T is a nontrivial tree with diameter at most 4, then $\gamma_{ve}(T) = 1$ and $\gamma_{ve}^t(T) = 2$.*

Observation 2.11. *Let G be a graph and $u \in V(G)$ a support vertex or a non-leaf vertex adjacent to an end-support vertex. If G' is the graph obtained from G by adding a vertex v attached to u, then $\gamma_{ve}^t(G) = \gamma_{ve}^t(G')$, $\gamma_{ve}(G) = \gamma_{ve}(G')$, and $\gamma_t(G) = \gamma_t(G')$.*

Proof. If G' is a star, then the results are immediate. Suppose G' is not a star. Clearly, any $\gamma_t(G)$-set (resp., $\gamma_t(G')$-set) containing no leaves, contains u, and so is a TDS of G' (resp., G) yielding $\gamma_t(G') \leq \gamma_t(G)$ (resp., $\gamma_t(G) \leq \gamma_t(G')$). Hence $\gamma_t(G') = \gamma_t(G)$.

Assume now that D is a $\gamma^t_{ve}(G')$-set containing no leaves. Then $u \in D$ if u is adjacent to an end-support vertex, and $D \cap N[u] \neq \emptyset$ when u is a support vertex. In both cases, D is clearly a total ve-dominating set of G, implying that $\gamma^t_{ve}(G') \geq \gamma^t_{ve}(G)$. On the other hand, any $\gamma^t_{ve}(G)$-set is a total ve-dominating set of G', and so $\gamma^t_{ve}(G') \leq \gamma^t_{ve}(G)$. Hence $\gamma^t_{ve}(G') = \gamma^t_{ve}(G)$. Similarly, we can see that $\gamma_{ve}(G') = \gamma_{ve}(G)$. □

3. Trees T with $\gamma^t_{ve}(T) = \gamma_t(T)$

In this section, we provide a constructive characterization of all trees T with $\gamma^t_{ve}(T) = \gamma_t(T)$. For this purpose, we define the family $\mathcal{T}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees such that $T_1 \in \{P_2, P_3, P_4\}$ and $T = T_k$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $u \in V(T_i)$ is a support vertex or a non-leaf vertex adjacent to an end-support vertex, then $\mathcal{O}_1$ adds a new vertex x and an edge ux to obtain T_{i+1}.

Operation $\mathcal{O}_2$: If $u \in W^1_{T_i}$, then $\mathcal{O}_2$ adds a path $P_4 = x_4x_3x_2x_1$ and an edge ux_1 to obtain T_{i+1}.

The next lemma is an immediate consequence of Observation 2.11.

Lemma 3.1. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_t(T_{i+1})$.*

Lemma 3.2. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_t(T_{i+1})$.*

Proof. By Proposition 2.5, we have $\gamma_t(T_{i+1}) = \gamma_t(T_i) + 2$. Assume next that D is a $\gamma^t_{ve}(T_{i+1})$-set such that $d(x_4, D)$ is as large as possible. Clearly, $x_1, x_2 \in D$, and thus $D \setminus \{x_1, x_2\}$ is an almost total ve-dominating set of T_i. We deduce from the assumption $u \in W^1_{T_i}$ that $\gamma^t_{ve}(T_{i+1}) \geq \gamma^t_{ve}(T_i; u) + 2 = \gamma^t_{ve}(T_i) + 2$. By Observation 2.6, we obtain $\gamma^t_{ve}(T_{i+1}) = \gamma_t(T_{i+1})$. □

Theorem 3.3. *If $T \in \mathcal{T}$, then $\gamma^t_{ve}(T) = \gamma_t(T)$.*

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 \in \{P_2, P_3, P_4\}$, and if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the aforementioned operations. We proceed by induction on the number of operations used to construct T. If $k = 1$, then $T \in \{P_2, P_3, P_4\}$ and clearly $\gamma^t_{ve}(T) = \gamma_t(T)$. Assume that the result holds for each tree of $\mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma^t_{ve}(T') = \gamma_t(T')$. Since $T = T_k$ is obtained from T' by Operation $\mathcal{O}_1$ or $\mathcal{O}_2$, we conclude from Lemmas 3.1 and 3.2 that $\gamma^t_{ve}(T) = \gamma_t(T)$. □

Now we are ready to state the main theorem of this section.

Theorem 3.4. *Let T be a tree of order $n \geq 2$. Then $\gamma^t_{ve}(T) = \gamma_t(T)$ if and only if $T \in \mathcal{T}$.*

Proof. According to Theorem 3.3, we need only to prove necessity. We proceed by induction on n. If $n \leq 3$, then $T \in \{P_2, P_3\}$ and clearly $T \in \mathcal{T}$. Let $n \geq 4$ and let the result hold for every tree T' of order less than n, satisfying $\gamma^t_{ve}(T') = \gamma_t(T')$. Let T be a tree of order n with $\gamma^t_{ve}(T) = \gamma_t(T)$. If $\mathrm{diam}(T) = 2$, then T is a star that can be obtained from P_3 by using Operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{p,q}$, $(q \geq p \geq 1)$ and we have $\gamma^t_{ve}(T) = \gamma_t(T)$. If $q = 1$, then $T = P_4 \in \mathcal{T}$, while if $q > 1$, then T can be obtained from P_4 by frequently use of $\mathcal{O}_1$, and thus $T \in \mathcal{T}$. Henceforth, we assume that $\mathrm{diam}(T) \geq 4$.

If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to x. By Observation 2.11 and the induction hypothesis, we have $T' \in \mathcal{T}$. Now T can be obtained from T' by operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$. Henceforth, we assume that T has no strong support vertex.

Let $v_1v_2\ldots v_k$ $(k \geq 5)$ be a diametrical path in T. Root T at v_k. Clearly, $\deg_T(v_2) = \deg_T(v_{k-1}) = 2$. Let D be a $\gamma_t(T)$-set containing no leaves. Clearly, such a set D exists since $\mathrm{diam}(T) \geq 4$. Moreover, D contains all support vertices of T. Hence $\{v_2, v_3\} \subseteq D$. We claim that v_4 does not belong to D. Indeed, if $v_4 \in D$, then $D - \{v_2\}$ is a total ve-dominating set of T of size $\gamma_t(T) - 1$, a contradiction. Therefore $v_4 \notin D$. In the following, we consider the following cases.

<u>*Case 1.*</u> $\deg_T(v_3) \geq 3$.
We claim that v_3 has no children with depth 1 different from v_2. Suppose, to the contrary, that y_2 is a child of v_3 with depth 1 and let $v_3y_2y_1$ be a pendant path in T. Let $T' = T - T_{v_2}$. Then $\{v_2, v_3, y_2\} \subseteq D$, and the set $D \setminus \{v_2\}$ is a TDS of T', implying that $\gamma_t(T) \geq \gamma_t(T') + 1$. Suppose now that S is a $\gamma^t_{ve}(T')$-set containing no leaves. Then clearly $v_3 \in S$, and so the set S is also a total ve-dominating set of T, yielding $\gamma^t_{ve}(T) \leq \gamma^t_{ve}(T')$. It follows from (1) and the assumption that

$$\gamma_t(T') \geq \gamma^t_{ve}(T') \geq \gamma^t_{ve}(T) = \gamma_t(T) \geq \gamma_t(T') + 1,$$

a contradiction.

Thus we may assume that v_3 has at least one children with depth 0, say x. Let $T' = T - \{x\}$. Clearly, $\gamma_t(T) \geq \gamma_t(T')$ since D remains a TDS of T'. Suppose now that S is a $\gamma^t_{ve}(T')$-set. To ve-dominate the edge v_1v_2, we must have $|\{v_2, v_3\} \cap S| \geq 1$ and hence S is a total ve-dominating set of T, yielding $\gamma^t_{ve}(T) \leq \gamma^t_{ve}(T')$. We conclude from Observation 2.7 that $\gamma^t_{ve}(T') = \gamma_t(T')$, and by the induction hypothesis, we have $T' \in \mathcal{T}$. Now T can be obtained from T' by operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$.

<u>*Case 2.*</u> $\deg_T(v_3) = 2$.
We claim that $\deg_T(v_4) = 2$. Suppose, to the contrary, that $\deg_T(v_4) \geq 3$. Observe that if v_4 has a child with depth 1 or 2, then $v_4 \in D$, contradicting the fact that v_4 does not belong to D. Hence every child of v_4 has depth 2. According to the Case 1, we can assume that every child of v_4 has degree two. Let z_3 be a child of

v_4 different from v_3, and let z_2 and z_1 be the children of z_3 and z_2, respectively. Clearly, $\{v_2, v_3, z_2, z_3\} \subseteq D$, but then $\{v_4\} \cup D \setminus \{v_2, z_2\}$ is a total ve-dominating set of T of size $\gamma_t(T) - 1$, a contradiction. Therefore, $\deg_T(v_4) = 2$. Now, let $T' = T - T_{v_4}$. Note that T' is nontrivial for otherwise $T = P_5$ and $\gamma_{ve}^t(P_5) < \gamma_t(P_5)$. We conclude from Proposition 2.5 and Observation 2.7 that $\gamma_{ve}^t(T') = \gamma_t(T')$. By induction on T', we have $T' \in \mathcal{T}$. Now let us show that $v_5 \in W_{T'}^1$. Suppose, to the contrary, that $v_5 \notin W_{T'}^1$, and let S' be an almost total ve-dominating set of T' with respect to v_5 of cardinality at most $\gamma_{ve}^t(T') - 1$. Then $S' \cup \{v_3, v_4\}$ is a total ve-dominating set of T with cardinality $\gamma_{ve}^t(T') + 1$, which is a contradiction. Hence $v_5 \in W_{T'}^1$. Now since T can be obtained from T' by Operation $\mathcal{O}_2$, we have $T \in \mathcal{T}$. This completes the proof. □

4. Graphs G with $\gamma_{ve}^t(G) = \gamma_{ve}(G)$

4.1. Hardness result

We show that it is NP-hard to decide whether $\gamma_{ve}^t(G) = \gamma_{ve}(G)$ for a given $(K_4 - e)$-graph G by reducing the 3-satisfiability problem (3-SAT problem) to our problem.

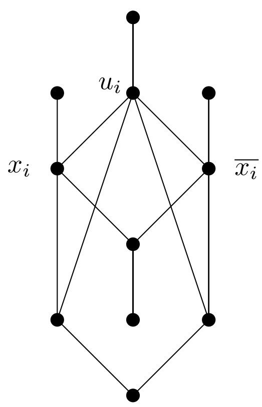

Figure 1. The graph H_i.

Theorem 4.1. *It is NP-hard to decide whether $\gamma_{ve}^t(G) = \gamma_{ve}(G)$ for a given $(K_4 - e)$-free graph G.*

Proof. Let $U = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We construct a graph G whose order is polynomially bounded in terms of n and m such that $\mathcal{C}$ is satisfiable if and only if $\gamma_{ve}^t(G) = \gamma_{ve}(G)$.

For each variable $x_i \in U$, associate the connected graph H_i as shown in Figure 1. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a path $P_2 = c_j w_j$.

For every literal $x \in \{x_i, \overline{x}_i\}$ and every clause C_j such that x appears in C_j, add an edge between c_j and the vertex denoted x in H_i. Clearly, G is $(K_4 - e)$-free. Also, for every ve-dominating set D of G, we have $|D \cap V(H_i)| \geq 2$, and thus $\gamma_{ve}(G) \leq 2n$. The equality is obtained from the fact that all x_i's and $\overline{x}_i$'s form a ve-dominating set of G. Furthermore, $\gamma^t_{ve}(G) = 2n$ holds if and only if every total ve-dominating set of G, that contains u_i and one vertex of $\{x_i, \overline{x}_i\}$ for every i. Clearly, such a total ve-dominating set of G indicates a satisfying truth assignment for $\mathcal{C}$. Moreover, from any satisfying truth assignment for $\mathcal{C}$, we can construct a total ve-dominating set of G of cardinality $2n$. Therefore, $\gamma^t_{ve}(G) = \gamma_{ve}(G)$ if and only if $\mathcal{C}$ is satisfiable. □

4.2. Trees T with $\gamma^t_{ve}(T) = \gamma_{ve}(T)$

In this subsection, we provide a constructive characterization of all trees T with $\gamma^t_{ve}(T) = \gamma_{ve}(T)$. For this purpose, we define the family $\mathcal{F}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees such that $T_1 \in \{P_6\}$ and $T = T_k$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{T}_1$: If $u \in V(T_i)$ is a support vertex or a non-leaf vertex adjacent to an end-support vertex, then $\mathcal{T}_1$ adds a new vertex x and an edge ux to obtain T_{i+1}.

Operation $\mathcal{T}_2$: If $u \in V(T_i)$ has degree at least two and is adjacent to an end-support vertex w, then $\mathcal{T}_2$ adds a path $P_2 = x_2x_1$ and an edge ux_2 to obtain T_{i+1}.

Operation $\mathcal{T}_3$: If $u \in V(T_i)$ is a leaf of an induced path $uvy_3y_2y_1$ such that $\deg_{T_i}(y_2) = \deg_{T_i}(y_3) = 2$ and $\deg_{T_i}(y_1) = 1$, then $\mathcal{T}_3$ adds a path $P_2 = x_2x_1$ and joins u to x_2 to obtain T_{i+1}.

Operation $\mathcal{T}_4$: If $u \in V(T_i)$ and there is a path $ux_3x_2x_1$ in T_i such that $\deg_{T_i}(x_2) = \deg_{T_i}(x_3) = 2$ and $\deg_{T_i}(x_1) = 1$, then $\mathcal{T}_4$ adds a new vertex y and an edge uy to obtain T_{i+1}.

Operation $\mathcal{T}_5$: If $u \in W^2_{T_i} \cap W^3_{T_i}$, then $\mathcal{T}_5$ adds a path $P_6 = x_6x_5x_4x_3x_2x_1$ and joins u to x_5 to obtain T_{i+1}.

Operation $\mathcal{T}_6$: If $u \in W^4_{T_i}$ is a leaf and its support vertex, say v, is adjacent to the center vertex of a pendant star $K_{1,s}$ centered at x, then $\mathcal{T}_6$ adds a path $P_3 = y_3y_2y_1$ and joins u to y_3 to obtain T_{i+1}.

Operation $\mathcal{T}_7$: If $u \in W^2_{T_i} \cap W^3_{T_i}$ and there exists a vertex $v \in N_{T_i}[u]$ such that $v \in W^4_{T_i}$, then $\mathcal{T}_7$ adds a path $P_6 = x_6x_5x_4x_3x_2x_1$ and an edge ux_6 to obtain T_{i+1}.

Operation $\mathcal{T}_8$: If $u \in V(T_i)$ such that $N_{T_i}(u) \cap W^4_{T_i} \neq \emptyset$ and uy_2y_1 is a pendant path in T_i, then $\mathcal{T}_8$ adds a path $P_3 = x_3x_2x_1$ and joins u to x_3 to obtain T_{i+1}.

Operation $\mathcal{T}_9$: If $u \in V(T_i)$, then $\mathcal{T}_9$ adds the graph H (see Figure 2) and joins u to v to obtain T_{i+1}.

Operation $\mathcal{T}_{10}$: If $u \in V(T_i)$ is adjacent to the vertex y_4 of a path $y_6y_5y_4y_3y_2y_1$ such that $\deg_{T_i}(y_4) = 3$, $\deg_{T_i}(y_6) = \deg_{T_i}(y_1) = 1$, and $\deg_{T_i}(y_5) = \deg_{T_i}(y_3) = \deg_{T_i}(y_2) = 2$, then $\mathcal{T}_{10}$ adds a path $P_6 = x_6x_5x_4x_3x_2x_1$ and joins u to x_4 to obtain T_{i+1}.

Operation $\mathcal{T}_{11}$: If $u \in V(T_i)$ and there is a path $y_5y_4uy_3y_2y_1$ in T_i such that $\deg_{T_i}(y_5) = \deg_{T_i}(y_1) = 1$ and $\deg_{T_i}(y_4) = \deg_{T_i}(y_3) = \deg_{T_i}(y_2) = 2$, then $\mathcal{T}_{11}$ adds a path $P_2 = x_2x_1$ and joins u to x_2 to obtain T_{i+1}.

Operation $\mathcal{T}_{12}$: If $u \in V(T_i)$ is adjacent to a support vertex of a pendant path P_6 and there exists a vertex $v \in N_{T_i}[u]$ such that $v \in W^4_{T_i}$, then $\mathcal{T}_{12}$ adds a new vertex x and an edge ux to obtain T_{i+1}.

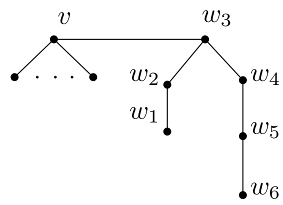

Figure 2. The graph H with $|L_v| \geq 0$ used in Operation $\mathcal{T}_9$.

The next result is an immediate consequence of Observation 2.11.

Lemma 4.2. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_1$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Lemma 4.3. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_2$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Clearly, any $\gamma^t_{ve}(T_i)$-set containing no leaves, contains u and so such a set remains a total ve-dominating set of T_{i+1}, implying that $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i)$. On the other hand, it is not hard to see that T_{i+1} has a $\gamma_{ve}(T_{i+1})$-set D containing u and not x_2. Thus D ve-dominates $E(T_i)$, yielding $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i)$. Now, by Observation 2.8, we have $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.4. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_3$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Let S be a $\gamma^t_{ve}(T_i)$-set containing no leaves. Clearly, $|S \cap \{v, y_3, y_2\}| = 2$ and the set $(S \setminus \{v, y_3, y_2\}) \cup \{u, v, y_3\}$ is a total ve-dominating set of T_{i+1}, implying that $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i) + 1$. On the other hand, it is not hard to see that T_{i+1} has a $\gamma_{ve}(T_{i+1})$-set D containing u, y_3, and so $D \setminus \{u\}$ is a ve-dominating set of T_i, yielding $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i) + 1$. Therefore, we conclude from Observation 2.8 that $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.5. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_4$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Clearly, any $\gamma^t_{ve}(T_i)$-set containing no leaves, contains u, x_3, and thus remains a total ve-dominating set of T_{i+1}, implying that $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i)$. Since every $\gamma_{ve}(T_{i+1})$-set that does not contain y is a ve-dominating set of T_i, we have $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i)$. Now, by Observation 2.8, we obtain $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.6. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_5$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Obviously, any $\gamma^t_{ve}(T_i)$-set can be extended to a total ve-dominating set of T_{i+1} by adding x_3, x_4, implying that $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i) + 2$. Assume now that D is a $\gamma_{ve}(T_{i+1})$-set containing no leaves. Without loss of generality, we may assume that $x_3 \in D$. If $|D \cap \{x_i \mid 1 \leq i \leq 6\}| \geq 2$, then $D \setminus \{x_i \mid 1 \leq i \leq 6\}$ is an almost ve-dominating set of T_i with respect to u, and so $\gamma_{ve}(T_i) = \gamma_{ve}(T_i; u) \leq \gamma_{ve}(T_{i+1}) - 2$ because of $u \in W^3_{T_i}$. Hence let $|D \cap \{x_i \mid 1 \leq i \leq 6\}| = 1$. To ve-dominate the edge x_5x_6, we must have $u \in D$, and thus $D \setminus \{x_3\}$ is a ve-dominating set of T_i containing u. Since $u \in W^2_{T_i}$, we deduce that $\gamma_{ve}(T_{i+1}) = |D| \geq \gamma_{ve}(T_i) + 2$. Now, by Observation 2.8, we have $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.7. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_6$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Since $u \in W^4_{T_i}$, let S be a $\gamma^t_{ve}(T_i)$-set containing u. Then $S \cup \{y_3\}$ is a total ve-dominating set of T_{i+1}, and thus $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i) + 1$. Now, let D be a $\gamma_{ve}(T_{i+1})$-set containing no leaves. Clearly $|D \cap \{v, x\}| \geq 1$ and $|D \cap \{y_2, y_3\}| = 1$. Without loss of generality, we assume that $v, y_3 \in D$. Then $D \setminus \{y_3\}$ is a ve-dominating set of T_i, implying that $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i) + 1$. It follows from Observation 2.8 that $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.8. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_7$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Since there exists a vertex $v \in N_{T_i}[u]$ with $v \in W^4_{T_i}$, let S be a $\gamma^t_{ve}(T_i)$-set containing v. Then $S \cup \{x_3, x_4\}$ is a total ve-dominating set of T_{i+1}, and thus $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i) + 2$. On the other hand, let D be a $\gamma_{ve}(T_{i+1})$-set. To ve-dominate the edges ux_6, x_ix_{i-1} for $2 \leq i \leq 6$, we must have $|D \cap \{x_1, x_2, x_3\}| \geq 1$ and $|D \cap \{u, x_i \mid 1 \leq i \leq 6\}| \geq 2$. If $D \cap \{x_6, u\} \neq \emptyset$, then $(D \setminus \{u, x_i \mid 1 \leq i \leq 6\}) \cup \{u\}$ is a ve-dominating set of T_i containing u, implying that $\gamma_{ve}(T_{i+1}) = |D| \geq \gamma_{ve}(T_i) + 2$ (because of $u \in W^2_{T_i}$). If $D \cap \{x_6, u\} = \emptyset$, then $D \setminus \{x_i \mid 1 \leq i \leq 6\})$ is a ve-dominating set of T_i, and so $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i) + 2$. Now, by Observation 2.8, we obtain $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.9. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_8$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Clearly, any $\gamma^t_{ve}(T_i)$-set S containing no leaves, must contain u, and so it can be extended to a total ve-dominating set of T_{i+1} by adding y_3, which implies that $\gamma^t_{ve}(T_{i+1}) \leq \gamma^t_{ve}(T_i) + 1$. On the other hand, for any $\gamma_{ve}(T_{i+1})$-set D containing no leaves, we have $|D \cap \{x_2, x_3\}| = 1$, and $|D \cap \{u, y_2\}| = 1$. Then $D \setminus \{x_2, x_3\}$ is a ve-dominating set of T_i, yielding $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i) + 1$. By Observation 2.8, we have $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.10. *If T_i is a tree with $\gamma^t_{ve}(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_9$, then $\gamma^t_{ve}(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Clearly, any $\gamma_{ve}^t(T_i)$-set can be extended to a total ve-dominating set of T_{i+1} by adding w_3, w_4, and so $\gamma_{ve}^t(T_{i+1}) \leq \gamma_{ve}^t(T_i) + 2$. On the other hand, let D be a $\gamma_{ve}(T_{i+1})$-set D. Then we must have $|D \cap \{w_1, w_2, w_3\}| \geq 1$, and $|D \cap \{w_4, w_5, w_6\}| \geq 1$. Without loss of generality, let $w_3, w_4 \in D$. If $|D \cap V(H)| \geq 3$, then $(D \setminus V(H)) \cup \{u\}$ is a ve-dominating set of T_i and if $|D \cap V(H)| = 2$, then $D \setminus V(H)$ is a ve-dominating set of T_i of size $\gamma_{ve}(T_{i+1}) - 2$. In any case, $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i) + 2$. It follows from Observation 2.8 that $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.11. *If T_i is a tree with $\gamma_{ve}^t(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_{10}$, then $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Clearly, any $\gamma_{ve}^t(T_i)$-set can be extended to a total ve-dominating set of T_{i+1} by adding x_3, x_4, and so $\gamma_{ve}^t(T_{i+1}) \leq \gamma_{ve}^t(T_i) + 2$. Now let D be a $\gamma_{ve}(T_{i+1})$-set. Without loss of generality, we may assume that $x_3, x_4, y_3, y_4 \in D$. Then $D \setminus \{x_3, x_4\}$ is a ve-dominating set of T_i, and so $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i) + 2$. By Observation 2.8, we obtain $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.12. *If T_i is a tree with $\gamma_{ve}^t(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_{11}$, then $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Clearly, any $\gamma_{ve}^t(T_i)$-set D such that $d(D, \{y_1, y_5\})$ is as large as possible, contains y_3, u, and so it is a total ve-dominating set of T_{i+1}, yielding $\gamma_{ve}^t(T_{i+1}) \leq \gamma_{ve}^t(T_i)$. Since there is a $\gamma_{ve}(T_{i+1})$-set that does not contain neither x_1 nor x_2, such a set is a ve-dominating set of T_i, and so $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i)$. By Observation 2.8, we obtain $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Lemma 4.13. *If T_i is a tree with $\gamma_{ve}^t(T_i) = \gamma_{ve}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_{12}$, then $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$.*

Proof. Since there exists a vertex $v \in N_{T_i}[u]$ with $v \in W_{T_i}^4$, let S be a $\gamma_{ve}^t(T_i)$-set containing v. Then S is a total ve-dominating set of T_{i+1}, and thus $\gamma_{ve}^t(T_{i+1}) \leq \gamma_{ve}^t(T_i)$. On the other hand, any $\gamma_{ve}(T_{i+1})$-set containing no leaf is a ve-dominating set of T_i, implying that $\gamma_{ve}(T_{i+1}) \geq \gamma_{ve}(T_i)$. Now, by Observation 2.8, we obtain $\gamma_{ve}^t(T_{i+1}) = \gamma_{ve}(T_{i+1})$. □

Theorem 4.14. *If $T \in \mathcal{F}$, then $\gamma_{ve}^t(T) = \gamma_{ve}(T)$.*

Proof. Let $T \in \mathcal{F}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 = P_6$, and if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the aforementioned operations. We proceed by induction on the number of operations used to construct T. If $k = 1$, then $T = P_6$ and clearly $\gamma_{ve}^t(T) = \gamma_{ve}(T)$. Assume that the result holds for each tree of $\mathcal{F}$ which can be obtained from a sequence of operations of length $k - 1$ and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_{ve}^t(T') = \gamma_{ve}(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{T}_i$ $(i = 1, 2, \ldots, 12)$ from T', we conclude from the Lemmas 4.2–4.13 that $\gamma_{ve}^t(T) = \gamma_{ve}(T)$. □

Now we are ready to state the main theorem of this section.

Theorem 4.15. *Let T be a tree of order $n \geq 6$. Then $\gamma^t_{ve}(T) = \gamma_{ve}(T)$ if and only if $T \in \mathcal{F}$.*

Proof. According to Theorem 4.14, we need only to prove necessity. Let T be a tree with $\gamma^t_{ve}(T) = \gamma_{ve}(T)$. By Observation 2.10, $\mathrm{diam}(T) \geq 5$, and so $n \geq 6$. We proceed by induction on n. If $n = 6$, then $T = P_6$ and clearly $T \in \mathcal{F}$. Let $n \geq 7$ and let the result hold for every tree T' of order less than n, satisfying $\gamma^t_{ve}(T') = \gamma_{ve}(T')$. Let T be a tree of order n with $\gamma^t_{ve}(T) = \gamma_{ve}(T)$. Let $v_1 v_2 \ldots v_k$ ($k \geq 6$) be a diametral path in T such that $\deg_T(v_2)$ is as large as possible. Among these paths, we choose a path such that $\deg_T(v_3)$ is as large as possible. Root T at v_k. If $\deg_T(v_2) \geq 3$, then let $T' = T - v_1$. By Observation 2.11 and the induction hypothesis, we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_1$. Henceforth, we assume that $\deg_T(v_2) = 2$. By the choice of diametrical path, we may assume that all end-support vertices on diametrical paths have degree two. In particular, any child of v_3 with depth 1 has degree 2.

First let $\deg_T(v_3) \geq 3$. We distinguish the following two situations:

- v_3 has a child y_2 with depth 1 different from v_2.
  Let $v_3 y_2 y_1$ be a pendant path in T, and let $T' = T - T_{v_2}$. Clearly, any $\gamma^t_{ve}(T)$-set D containing no leaves, contains v_3, and thus total ve-dominates $E(T')$, yielding $\gamma^t_{ve}(T') \leq \gamma^t_{ve}(T)$. On the other hand, if S is a $\gamma_{ve}(T')$-set such that $d(v_1, S)$ is as large as possible, then clearly $v_3 \in S$, and so S is a ve-dominating set of T', implying that $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. It follows form Observation 2.9 and the induction hypothesis $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it is obtained from T' by Operation $\mathcal{T}_2$.
- All children of v_3 but v_2 are leaves.
  Let x be a leaf adjacent to v_3 and let $T' = T - x$. By Observation 2.11 and the induction hypothesis, we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_1$.

From now on, we assume that $\deg_T(v_3) = 2$. Recall that by the choice of the diametrical path, we may assume that all children of v_4 with depth 2, have degree two. Also, according to the above cases, we may assume that $\deg(v_{k-1}) = \deg(v_{k-2}) = 2$. We consider the following cases:

<u>*Case 1.*</u> $\deg_T(v_4) \geq 3$.
We distinguish the following subcases.

<u>*Subcase 1.1.*</u> v_4 has a child z_3 with depth 2, different from v_3.
Let $v_4 z_3 z_2 z_1$ be a pendant path in T and let $T' = T - T_{v_2}$. Assume that D is a $\gamma^t_{ve}(T)$-set such that $d(D, \{v_1, z_1\})$ is maximum. Then clearly $\{v_3, z_3, v_4\} \subseteq D$, and so $D \setminus \{v_3\}$ is a total ve-dominating set of T', implying that $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 1$. On the other hand, any $\gamma_{ve}(T')$-set can be extended to a ve-dominating set of T' by adding v_2, yielding $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. We deduce from Observation 2.9 and the induction hypothesis that $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ since it is obtained from T' by Operation $\mathcal{T}_3$.

Subcase 1.2. v_4 has a child y_2 with depth 1 and degree at least 3.
Let x be a leaf adjacent to y_2 and let $T' = T - x$. By Observation 2.11 and the induction hypothesis, we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ because it is obtained from T' by Operation $\mathcal{T}_1$.

Subcase 1.3. v_4 has a child with depth 0.
Let y be a leaf adjacent to v_4 and let $T' = T - y$. If D is a $\gamma^t_{ve}(T)$-set such that $d(D, v_1)$ is maximum, then clearly $\{v_3, v_4\} \subseteq D$. Hence D is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T')$. On the other hand, if S is a $\gamma_{ve}(T')$-set such that $d(S, v_1)$ is maximum, then $v_3 \in S$, and thus S ve-dominates $E(T)$. Hence $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. By Observation 2.9 and the induction hypothesis we have $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it is obtained from T' by Operation $\mathcal{T}_4$.

Subcase 1.4. $\deg_T(v_4) \geq 4$ and any child of v_4 is of depth 1 and degree 2.
Let $v_4y_2y_1$ and $v_4z_2z_1$ be two pendant paths in T and let $T' = T - \{z_2, z_1\}$. Clearly, any $\gamma_{ve}(T')$-set D such that $d(D, \{y_1, v_1\})$ is as large as possible, contains v_3, v_4, and so D ve-dominates $E(T)$, yielding $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. On the other hand, any $\gamma^t_{ve}(T)$-set D such that $d(D, \{y_1, v_1\})$ is as large as possible, contains v_3, v_4, and so it is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T')$. By Observation 2.9 and the induction hypothesis, we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_{11}$.

Subcase 1.5. $\deg_T(v_4) = 3$ and v_4 has exactly one child with depth 1 and degree 2.
Let $v_4y_2y_1$ be a pendant path in T. We distinguish the following.

(a) v_5 is a support vertex or $\deg_T(v_5) = 2$.
Assume that D is a $\gamma^t_{ve}(T)$-set. Clearly $\{v_3, v_4\} \subseteq D$. Suppose first that $v_5 \in D$, and let $T' = T - T_{v_3}$. Clearly, $D \setminus \{v_3\}$ is a total ve-dominating set of T', implying that $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 1$. On the other hand, if S is a $\gamma_{ve}(T')$-set, then $S \cup \{v_3\}$ is a ve-dominating set of T, implying that $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. By Observation 2.9 and the induction hypothesis, we obtain $T' \in \mathcal{F}$, where $D \setminus \{v_3\}$ is $\gamma^t_{ve}(T')$-set containing v_5, that is, $v_5 \in W^4_{T'}$. It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_8$. Suppose now that $v_5 \notin D$, and let $T' = T - T_{v_5}$. Note that, we can assume that v_5 has no child with depth 1 or 2 for otherwise v_5 belongs to some $\gamma^t_{ve}(T)$-set, and such a case was already considered. Moreover, if v_5 has a child with depth 3, then this situation is considered more generally in items (c) and (d). Hence we can assume that every child of v_5 besides v_4 (if any) is a leaf. Also we note that if $k = 6$, then T is isomorphic to H that belongs to $\mathcal{F}$ (it can be obtained from T_1 by using Operation $\mathcal{T}_2$ and possibly Operation $\mathcal{T}_1$) . Hence we assume that T' is nontrivial. Obviously, $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. Also, if S is a $\gamma_{ve}(T')$-set, then $S \cup \{v_3, v_4\}$ is a ve-dominating set of T, implying that $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 2$. By Observation 2.9 and the induction hypothesis, we have $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_9$.

(b) $\deg_T(v_5) \geq 3$ and v_5 has a child with depth 1 or 2.
Let $T' = T - T_{v_3}$. It is not hard to see that $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 1$, $\gamma_{ve}(T) \leq$

$\gamma_{ve}(T')+1$, and v_5 belongs to some $\gamma^t_{ve}(T')$-set. We deduce from Observation 2.9 and the induction hypothesis that $T' \in \mathcal{F}$, where $v_5 \in W^4_{T'}$. Therefore, $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_8$.

(c) $\deg_T(v_5) \geq 3$ and v_5 has a child y_4 with depth 3 and degree at least 3. Let $v_5y_4y_3y_2y_1$ be a path in T. Then $v_k \ldots v_5y_4y_3y_2y_1$ is a diametral path in T and by the assumption, we have $\deg(y_2) = \deg(y_3) = 2$. Also, according to the above cases and subcases above, we have T_{y_4} is isomorphic to T_{v_4}. Let $T' = T - T_{v_4}$. Clearly for every $\gamma^t_{ve}(T)$-set D containing v_3, v_4, y_3, y_4, we have $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T')+2$. Also, if S is a $\gamma_{ve}(T')$-set, then $S \cup \{v_3, v_4\}$ is a ve-dominating set of T, and so $\gamma_{ve}(T) \leq \gamma_{ve}(T')+2$. It follows from Observation 2.9 and the induction hypothesis that $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it can be obtained from T' by Operation $\mathcal{T}_{10}$.

(d) $\deg_T(v_5) \geq 3$ and all children of v_5 of depth 3, but v_4 have degree two. Note that v_5 can be a support vertex. Let $T' = T - T_{v_5}$. It is easy to see that $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2\deg_T(v_5) - 2$ and $\gamma_{ve}(T) \leq \gamma_{ve}(T') + \deg_T(v_5)$. This leads to $\gamma_{ve}(T) \leq \gamma^t_{ve}(T) - \deg_T(v_5) + 2 < \gamma^t_{ve}(T)$, a contradiction.

Case 2. $\deg_T(v_4) = 2$.
Considering the arguments above, we may assume that for any diametrical path $z_1z_2 \ldots z_k$ in T, $\deg_T(z_i) = 2$ for all $i \in \{2, 3, 4, k-1, k-2, k-3\}$. Since $n \geq 7$, it follows that $\mathrm{diam}(T) \geq 6$. Consider the following subcases:

Subcase 2.1. v_5 has at least two children with depth 0.
Let $\{x, y\} \subseteq L_{v_5}$ and let $T' = T - x$. By Observation 2.11, we have $\gamma^t_{ve}(T') = \gamma_{ve}(T')$, and by the induction hypothesis, $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it can be obtained from T' by Operation $\mathcal{T}_1$.

Subcase 2.2. v_5 has a child with depth 1.
Let $T' = T - T_{v_3}$. Clearly, v_3, v_4, v_5 belong to any $\gamma^t_{ve}(T)$-set, and so $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 1$. Also, if S is a $\gamma_{ve}(T')$-set, then $S \cup \{v_3\}$ is a ve-dominating set of T, and thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. It follows from Observation 2.9 and the induction hypothesis that $T' \in \mathcal{F}$, where in particular, $v_4 \in W^4_{T'}$ (since it has a child with depth 1). It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_6$.

Subcase 2.3. v_5 has a child y_3 with depth 2.
Let $v_5y_3y_2y_1$ be a path in T and let $T' = T - T_{v_4}$. Clearly, $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Also if D is a $\gamma^t_{ve}(T)$-set containing no leaves, then D must contain v_3, v_4, y_3, v_5, and so $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T'. Hence $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T')+2$, and thus

$$\gamma_{ve}(T) = \gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2 \geq \gamma_{ve}(T') + 2 \geq \gamma_{ve}(T) + 1,$$

a contradiction.

Subcase 2.4. v_5 has a child z_4 with depth 3.
Let $v_5z_4z_3z_2z_1$ be a path in T and let $T' = T - T_{v_4}$. According to cases above, $\deg_T(z_i) = 2$ for $i \in \{2, 3, 4\}$. Now, if D is a $\gamma^t_{ve}(T)$-set such that $d(D, \{v_1, z_1\})$ is as large as possible, then $\{v_3, z_3, v_4, z_4\} \subseteq D$, and so $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. Also, it is easy to see

that $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. These two inequalities lead to a contradiction as in Subcase 2.3.

Subcase 2.5. $\deg_T(v_5) = 3$ and v_5 has one child w with depth 0.

- v_6 is a support vertex.

Let $z \in L_{v_6}$ and let $T' = T - z$. If D is a $\gamma^t_{ve}(T)$-set containing no leaves, then clearly D contains a vertex $N_T[v_6]$, and so D remains a total ve-dominating set of T'. Hence $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T')$. Also, since there exists a $\gamma_{ve}(T')$-set S containing at least one vertex of $N_{T'}[v_6]$, we have $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. By Observation 2.9, $\gamma_{ve}(T') = \gamma^t_{ve}(T')$, where some vertex of $N_{T'}[v_6]$ belongs to $W^4_{T'}$. It follows from the induction hypothesis that $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_{12}$.

- v_6 is not a support vertex.

Let $T' = T - T_{v_5}$. Assume that D is a $\gamma^t_{ve}(T)$-set. Clearly $\{v_3, v_4\} \subseteq D$. If $v_5 \notin D$, then $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', and so $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. If $v_5 \in D$ and $v_6 \in D$, then $(D \setminus \{v_3, v_4, v_5\}) \cup \{v_7\}$ is a total ve-dominating set of T'. If $v_5 \in D$ and $v_6 \notin D$, then $N_{T'}[v_7] \notin D$, for otherwise $D \setminus \{v_5\}$ is a total ve-dominating set of T with cardinality less than $|D|$, a contradiction. Hence $D \cap (N(v_8) - \{v_7\}) \neq \emptyset$ to total ve-dominate edge v_7v_8, and thus $(D \setminus \{v_5\}) \cup \{v_8\}$ is a total ve-dominating set of T'. In any case, we have $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$.

Now let S be a $\gamma_{ve}(T')$-set. If $v_6 \in S$, then $S \cup \{v_3\}$ is a ve-dominating set of T, and thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Hence

$$\gamma_{ve}(T) = \gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2 \geq \gamma_{ve}(T') + 2 \geq \gamma_{ve}(T) + 1,$$

a contradiction. Hence $v_6 \notin S$, and more gererally, $v_6 \in W^2_{T'}$. Thus $S \cup \{v_3, v_5\}$ is a ve-dominating set of T, implying that $\gamma_{ve}(T) \leq \gamma_{ve}(T')+2$. By Observation 2.9, we have $\gamma_{ve}(T') = \gamma^t_{ve}(T')$, and thus $\gamma^t_{ve}(T) = \gamma^t_{ve}(T')+2$ and $\gamma_{ve}(T) = \gamma_{ve}(T')+2$. It follows from the induction hypothesis that $T' \in \mathcal{F}$. We prove now that $v_6 \in W^3_{T'}$. Suppose, to the contrary, that $v_6 \notin W^3_{T'}$ and let S' be an almost ve-dominating set of T' with respect to v_6 of cardinality at most $\gamma_{ve}(T') - 1$. Hence $S' \cup \{v_3, v_5\}$ is a ve-dominating set of T, and so $\gamma_{ve}(T) \leq \gamma_{ve}(T'; v_6) + 2 \leq \gamma_{ve}(T') + 1$, a contradiction. Therefore, $v_6 \in W^3_{T'}$, and so $v_6 \in W^2_{T'} \cap W^3_{T'}$. It follows that $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_5$.

Subcase 2.6. $\deg(v_5) = 2$.

According to the previous cases above, we may assume that for any diametrical path $z_1z_2 \ldots z_k$ in T, $\deg(z_i) = 2$ for every $i \in \{2, 3, 4, 5, z_{k-4}, z_{k-3}, z_{k-2}, z_{k-1}\}$. We distinguish the following subcases:

(i) v_6 has a child y_2 with depth 1 or a child y_3 with depth 2 or a child y_4 with depth 3.

Let $v_6y_ix_{i-1} \ldots x_1$ be a path in T, where $i \in \{2, 3, 4\}$, and let $T' = T - T_{v_4}$. Clearly, $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. If D is a $\gamma^t_{ve}(T)$-set such that $d(D, x_1)$ is maximum, then $v_3, v_4 \in D$ and $D \cap \{v_6, y_i\} \neq \emptyset$. Hence $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', and so $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. This leads to a contradiction as in Subcase 2.3.

(ii) v_6 has a child z_5 with depth 4, different from v_5.

Let $v_6z_5z_4z_3z_2z_1$ be a pendant path in T and let $T' = T - T_{v_5}$. Since there is a $\gamma^t_{ve}(T)$-set D containing v_3, z_3, v_4, z_4, $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', implying that $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. Also, since there is a $\gamma_{ve}(T')$-set S that contains a vertex of $N_{T'}[v_6]$ (because of the edge z_5v_6), then $S \cup \{v_3\}$ is a ve-dominating set of T, and so $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. As in Subcase 2.3, this leads to a contradiction.

(iii) $\deg(v_6) \geq 3$ and any child of v_6 is of depth 0.
Let $T' = T - T_{v_4}$. Then $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma^t_{ve}(T)$-set such that $d(v_1, D)$ is maximum. Then $\{v_3, v_4\} \subseteq D$. If $v_5 \notin D$ or $v_5, v_6 \in D$, then $D \setminus \{v_3, v_4\}$ is a total ve-dominating set of T', and hence $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. This situation leads to a contradiction as above. Hence assume that $v_5 \in D$ and $v_6 \notin D$. Let $T'' = T - T_{v_6}$. Then $D - \{v_3, v_4, v_5\}$ is a total ve-dominating set of T'', and so $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T'')+3$. Also, it is easy to see that $\gamma_{ve}(T) \leq \gamma_{ve}(T'')+2$, which leads to a contradiction as above.

(iv) $\deg_T(v_6) = 2$.
Since $\gamma^t_{ve}(T) = \gamma_{ve}(T)$, we have $\operatorname{diam}(T) \geq 7$. Let $T' = T - T_{v_6}$. Then $\gamma_{ve}(T) \leq \gamma_{ve}(T')+2$. Let D be a $\gamma^t_{ve}(T)$-set such that $d(D, v_1)$ is maximum. Then $v_3, v_4 \in D$. If $v_5 \in D$ and $v_6 \notin D$, then $D \setminus \{v_3, v_4, v_5\}$ is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 3$, and this leads to a contradiction as above. If $v_5, v_6 \notin D$, then D contains a vertex of $N_{T'}[v_7]$, and thus $D - \{v_3, v_4\}$ is a total ve-dominating set of T', and so $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. If $v_5, v_6 \in D$, then $(D \setminus \{v_3, v_4, v_5, v_6\}) \cup \{v_7, v_8\}$ is a total ve-dominating set of T', and so $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. If $v_5 \notin D$ and $v_6 \in D$, then $v_7 \in D$, and so $(D \setminus \{v_3, v_4, v_6\}) \cup \{v_8\}$ is a total ve-dominating set of T', yielding $\gamma^t_{ve}(T) \geq \gamma^t_{ve}(T') + 2$. By Observation 2.9, we obtain $\gamma^t_{ve}(T') = \gamma_{ve}(T')$, implying also $\gamma^t_{ve}(T) = \gamma^t_{ve}(T')+2$ and $\gamma_{ve}(T) = \gamma_{ve}(T') + 2$. By the induction hypothesis, $T' \in \mathcal{T}$. Next we show that $v_7 \in W^2_{T'} \cap W^3_{T'}$. If $v_7 \notin W^2_{T'}$, then any $\gamma_{ve}(T')$-set containing v_7 can be extended to a ve-dominating set of T by adding v_3, which leads to a contradiction (since $\gamma_{ve}(T) = \gamma_{ve}(T')+2$). If $v_7 \notin W^3_{T'}$, then any almost total ve-dominating set of T' of weight less than $\gamma_{ve}(T')$ can be extended to a ve-dominating set of T by adding v_3, v_6, which leads to a contradiction too. Hence $v_7 \in W^2_{T'} \cap W^3_{T'}$. Note that any total ve-dominating set of T' defined above, contains a vertex of $N_{T'}[v_7]$ and is a $\gamma^t_{ve}(T')$-set, that is $N_{T'}[v_7] \cap W^4_{T'} \neq \emptyset$. Therefore, $T \in \mathcal{F}$ since it can be obtained from T' by Operation $\mathcal{T}_7$.

This completes the proof. □

References

1. Abdollahzadeh Ahangar H., Amjadi J., Chellali M., Nazari-Moghaddam S., Chellali M. and Sheikholeslami S. M., *Total Roman reinforcement in graphs*, Discuss. Math. Graph Theory **39** (2019), 787–803.
2. Abdollahzadeh Ahangar H., Amjadi J., Jafari Rad N. and Samodivkin V., *Total k-rainbow domination numbers in graphs*, Commun. Comb. Optim. **3** (2018), 37–50.
3. Abdollahzadeh Ahangar H., Chellali M., Sheikholeslami S. M. and Valenzuela-Tripodoro J. C., *Total Roman {2}-dominating functions in graphs*, Discuss. Math. Graph Theory, to appear.

4. Boutrig R. and Chellali M., *Total vertex-edge domination*, Int. J. Comput. Math. **95** (2018), 1820–1828.
5. Boutrig R., Chellali M., Haynes T. W. and Hedetniemi S. T., *Vertex-edge domination in graphs*, Aequationes Math. **90** (2016), 355–366.
6. Cockayne E. J., Dawes R. M. and Hedetniemi S. T., *Total domination in graphs*, Networks **10** (1980), 211–219.
7. Henning M. A. and Yeo A., *Total Domination in Graphs*, Springer Monographs in Mathematics, 2013.
8. Krishnakumari B., Venkatakrishnan Y. B. and Krzywkowski M., *Bounds on the vertex-edge domination number of a tree*, C. R. Acad. Sci. Paris, Ser.I **352** (2014), 363–366.
9. Lewis J. R., *Vertex-edge and Edge-vertex Domination in Graphs*, Ph.D. Thesis, Clemson University, 2007.
10. Lewis J. R., Hedetniemi S. T., Haynes T. W. and Fricke G. H., *Vertex-edge domination*, Util. Math. **81** (2010), 193–213.
11. Miao F., Fan W., Chellali M., Khoeilar R., Sheikholeslami S. M. and Soroudi M., *On two open problems on double vertex-edge domination in graphs*, Mathematics **7** (2019), Art. ID 1010.
12. Peters J. W., *Theoretical and Algorithmic Results on Domination and Connectivity*, Ph.D. Thesis, Clemson University, 1986.

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