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ON MEAN STRETCH CURVATURES OF FINSLER METRICS

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ABSTRACT. In this paper, we prove that every compact Finsler manifold with positive (or negative) relatively isotropic mean stretch curvature is a weakly Landsberg metric. Then, we show that weakly stretch Finsler surface has vanishing $\tilde{\mathbf{B}}$-curvature if and only if it has vanishing $\mathbf{H}$-curvature.

1. INTRODUCTION

Let (M, F) be a Finsler manifold. The third order derivatives of $\frac{1}{2}F_x^2$ at non-zero vector $y \in T_xM_0$ is called the Cartan torsion $\mathbf{C}_y$ of F. The rate of change of $\mathbf{C}$ along Finslerian geodesics is the Landsberg curvature $\mathbf{L}$ of F. The Finsler metric F satisfying $\mathbf{L} = 0$ is called a Landsberg metric. As an meaningful extension of Landsberg curvature $\mathbf{L}$, Berwald [**2**](presented a new non-Riemannian quantity called by the stretch curvature $\mathbf{\Sigma}_y$ of F. For a non-zero vector $y \in T_xM_0$, define the stretch curvature $\mathbf{\Sigma}_y : T_xM \times T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(q, u, v, w) := \Sigma_{\ ijkl}(y)q^iu^jv^kw^l$, where

$$\Sigma_{ijkl} := L_{ijk|l} - L_{ijl|k}, \tag{1}$$

where "|" is the horizontal derivation with respect to the Berwald connection of F. The family $\mathbf{\Sigma} := \{\mathbf{\Sigma}_y\}_{y \in TM_0}$ is called the stretch curvature. F is called a stretch metric if $\mathbf{\Sigma} = 0$. As a geometric meaning, Berwald showed that the stretch curvature of F satisfies $\mathbf{\Sigma} = 0$ if and only if the length of an arbitrary vector is unchanged under the parallel displacement along an infinitesimal parallelogram. F is said to be stretch metric whenever $\mathbf{\Sigma} = 0$. Then, this curvature was studied by researchers such as Shibata [**9**], Matsumoto [**5**], Tayebi-Tabatabaeifar [**22**], and Tayebi-Najafi in [**15**]. In [**20**], Tayebi-Sadeghi showed that a regular (α, β)-metric of non-Randers type satisfying $\mathbf{S} = 0$ is a stretch metric if and only if it is a Berwald metric. Let F be an almost regular non-Randers type (α, β)-metric. Suppose that F is not Berwaldian. They found a family of stretch (α, β)-metrics which are not Landsberg metrics [**20**].

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In [**6**], Najafi-Tayebi introduced a new non-Riemannian quantity named mean stretch curvature. Taking trace with respect to $\mathbf{g}_y$ in first and second variables of $\mathbf{\Sigma}_y$ gives rise the mean stretch curvature $\bar{\mathbf{\Sigma}}_y$. A Finsler metric is said to be *weakly stretch metric* if $\bar{\mathbf{\Sigma}} = 0$. F satisfying $\bar{\mathbf{\Sigma}} = 0$ is called weakly stretch metric. The class of weakly stretch metric metrics contains the class of stretch metrics. Najafi-Tayebi [**6**] proved that every compact weakly stretch manifold with negative flag curvature reduces to a Riemannian manifold. A Finsler metric F on a manifold M is said to be of relatively isotropic mean stretch curvature if it satisfies

$$\bar{\Sigma}_{ij} = \lambda F(I_{i|j} - I_{j|i}), \tag{2}$$

where $\lambda = \lambda(x, y)$ is a scalar function on TM. If $\lambda \geq 0$ ($\lambda \leq 0$ or $\lambda =$ constant), then F is said to be of positive (negative or constant) relatively isotropic mean stretch curvature. It is obvious that every weakly stretch metric is of relatively isotropic mean stretch curvature $\lambda = 0$.

Example 1. A Finsler metric F satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. Let $\langle , \rangle$ and $|\cdot|$ be the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. The standard Funk metric on the Euclidean unit ball $\mathbb{B}^n(1)$ is defined by

$$\Theta(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2}, \qquad y \in T_x\mathbb{B}^n(1) \simeq \mathbb{R}^n, \tag{3}$$

F is of constant relatively isotropic mean stretch curvature $\lambda = -1/2$ (see [**18**]).

In [**6**], Najafi-Tayebi showed that every complete weakly stretch Finsler manifold with bounded mean Cartan torsion is a weakly Landsberg manifold. Thus, a compact weakly stretch Finsler manifold reduces to a weakly Landsberg manifold. In this paper, we generalize their result as follows.

Theorem 1.1. *Every compact Finsler manifold with positive (or negative) relatively isotropic mean stretch curvature is weakly Landsberg manifold. More precisely, every complete Finsler metric with constant relatively isotropic mean stretch curvature and bounded mean Landsberg curvature is a weakly Landsberg metric.*

In [**1**], Akbar-Zadeh defined the important and significant non-Riemannian quantity $\mathbf{H}$. The quantity $\mathbf{H}$ arises from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics. In the class of Finsler metrics of scalar flag curvature, vanishing $\mathbf{H}$-curvature results that the metric is of constant flag curvature [**7, 23**]. Similarly, Shen defined $\widetilde{\mathbf{B}}$-curvature which is obtained from the Berwald curvature $\mathbf{B}$ by the covariant horizontal differentiation along geodesics (see [**8**, page 138]). Then, every Finsler metric with vanishing $\widetilde{\mathbf{B}}$-curvature has vanishing $\mathbf{H}$-curvature. But the converse might not hold. In this paper, we find a condition on 2-dimensional Finsler metrics under which the converse of the mentioned problem holds. More precisely, we prove the following.

Theorem 1.2. *Let (M, F) be a weakly stretch Finsler surface. Then $\widetilde{\mathbf{B}} = 0$ if and only if $\mathbf{H} = 0$.*

2. Preliminaries

A Finsler metric on an n-dimensional manifold M is a function $F\colon TM \to [0,\infty)$ such that: (i) F is C^∞ on the slit tangent bundle $TM_0 = TM \setminus \{0\}$, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial r \partial s}\Big[F^2(y+ru+sv)\Big]\Big|_{r,s=0}, \qquad u,v \in T_xM.$$

The Cartan tensor $\mathbf{C}_y\colon T_xM \times T_xM \times T_xM \to \mathbb{R}$ is defined by $\mathbf{C}_y(u,v,w) := C_{ijk}(y)u^iv^jw^k$, where

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\Big[\mathbf{g}_{y+tw}(u,v)\Big]\Big|_{t=0}, \qquad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian [**21**].

Let (M,F) be a Finsler manifold. For $y \in T_xM_0$, define $\mathbf{I}_y\colon T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. Every positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I}_y = 0$.

The Landsberg tensor $\mathbf{L}_y\colon T_xM \times T_xM \times T_xM \to \mathbb{R}$ is defined by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$, and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is said the Landsberg curvature of F. F is called a Landsberg metric if $\mathbf{L} = 0$. The mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ is defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

For a vector $y \in T_xM$, the Landsberg and mean Landsberg curvature of F can be defined by

$$\mathbf{L}_y(u,v,w) := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big[\mathbf{C}_{\dot\sigma(t)}\Big(U(t),V(t),W(t)\Big)\Big]\Big|_{t=0}, \quad \mathbf{J}_y(u) := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big[\mathbf{I}_{\dot\sigma(t)}\Big(U(t)\Big)\Big]\Big|_{t=0},$$

where $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$, and $U(t), V(t), W(t)$ are three linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the Landsberg (resp., mean Landsberg) curvature measures the rate of change of the Cartan (resp., mean Cartan) torsion along Finslerian geodesics [**10, 16**].

For $y\in T_xM_0$, define $\bar{\mathbf{\Sigma}}_y\colon T_xM\times T_xM\to\mathbb{R}$ by $\bar{\mathbf{\Sigma}}_y(u,v):=\bar{\Sigma}_{ij}(y)u^iv^j$, where $\bar{\Sigma}_{ij}:=g^{kl}\Sigma_{klij}$. In local coordinate, it is defined by

$$\bar{\Sigma}_{ij}=2(J_{i|j}-J_{j|i}).$$

F is called a weakly stretch metric if it satisfies $\bar{\mathbf{\Sigma}}=0$.

For a Finsler manifold (M,F), one can define a global vector field $\mathbf{G}$ which is induced by F on TM_0 and to be said the spray associated to (M,F). In a standard coordinate (x^i,y^i) for TM_0, $\mathbf{G}$ is defined by $\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i(x,y):=\frac{1}{4}g^{il}(y)\left\{\frac{\partial^2[F^2]}{\partial x^k\partial y^l}(x,y)y^k-\frac{\partial[F^2]}{\partial x^l}(x,y)\right\},\qquad y\in T_xM,$$

are local functions on TM.

For a tangent vector $y\in T_xM_0$, define $\mathbf{B}_y\colon T_xM\times T_xM\times T_xM\to T_xM$ and $\mathbf{E}_y:T_xM\otimes T_xM\to\mathbb{R}$ by $\mathbf{B}_y(u,v,w):=B^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u,v):=E_{jk}(y)u^jv^k$, where

$$B^i{}_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}\qquad\text{and}\qquad E_{jk}:=\frac{1}{2}B^m{}_{jkm}.$$

The non-Riemannian quantities $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature of F, respectively. F is a Berwald (resp. weakly Berwald) metric if it satisfies $\mathbf{B}=0$ (resp., $\mathbf{E}=0$) [**11, 12, 13, 14**].

Let us define $\widetilde{\mathbf{B}}_y\colon T_xM\times T_xM\times T_xM\to T_xM$ and $\mathbf{H}_y:T_xM\times T_xM\to\mathbb{R}$ by $\widetilde{\mathbf{B}}_y(u,v,w):=\widetilde{B}^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{H}_y(u,v):=H_{ij}(y)u^iv^j$, where

$$\widetilde{B}^i{}_{jkl}:=B^i{}_{jkl|s}y^s\qquad\text{and}\qquad H_{ij}:=E_{ij|s}y^s.$$

Then $\widetilde{\mathbf{B}}_y$ and $\mathbf{H}_y$ are defined as the covariant derivative of $\mathbf{B}$ and $\mathbf{E}$ along geodesics, respectively, [**7, 19, 23**].

Let (M,F) be an n-dimensional Finsler manifold and fix a local frame $\{\mathbf{b}_i\}$ for TM. Let lift the local frame $\{\mathbf{b}_i\}$ to a local frame $\{\mathbf{e}_i\}$ for the pull-back tangent bundle π^*TM by setting $\mathbf{e}_i(x,y):=(y,\mathbf{b}_i(x))$. Let $\{\omega^i,\omega^{n+i}\}$ denote the corresponding local coframe for $T^*(TM_0)$. The Berwald connection forms are the unique local 1-forms $\{\omega_j{}^i\}$ satisfying

$$\begin{aligned}\mathrm{d}\,\omega^i&=\omega^j\wedge\omega_j{}^i,\\ \mathrm{d}\,g_{ij}&=g_{ik}\omega_j{}^k+g_{kj}\omega_i{}^k-2L_{ijk}\omega^k+2C_{ijk}\omega^{n+k},\end{aligned}$$

where $\omega^{n+k}:=\mathrm{d}\,y^k+y^j\omega_j^k$ and y^i are viewed as local functions on TM, whose values y^i at y are defined by $y=y^i\mathbf{b}_i$.

In this paper, we use the Berwald connection of Finsler metrics, and the h- and v- covariant derivatives of a tensor field are denoted by "|" and ",", respectively.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. In order to prove it, we remark the following.

Theorem 3.1 ([**4**])**.** *Let M be a orientable manifold with volume form ω and ∇ be a torsion-free connection on M such that $\nabla\omega = 0$. Then for every vector field X and $v \in T_xM$, $x \in M$, the following holds*

$$(\operatorname{Div} X)_x = -\operatorname{trace}(X \mapsto \nabla_v X) = \nabla_i X^i.$$

Also, the following theorem holds.

Theorem 3.2 ([**4**])**.** *Let M be an orientable manifold with volume form ω and ∇ be a torsion-free connection on M such that $\nabla\omega = 0$. Then for every vector field X on M, the following holds*

$$\int_M (\operatorname{Div} X)\omega = 0.$$

Now, we can prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Let $p \in M$, and $y, u, v, w \in T_pM$. Let $c\colon (-\infty, \infty) \to M$ be the unit speed geodesic passing from p and

$$\frac{\mathrm{d}\, c}{\mathrm{d}\, t}(0) = y.$$

Suppose that $U = U(t)$, $V = V(t)$ and $W = W(t)$ are the parallel vector fields along c with $U(0) = u, V(0) = v$, and $W(0) = w$. Let us put

$$\mathbf{J}(t) = \mathbf{J}\big(U(t), V(t), W(t)\big) \qquad \text{and} \qquad \mathbf{J}'(t) = \mathbf{J}'\big(U(t), V(t), W(t)\big).$$

By assumption, F has positive (negative or constant) relatively isotropic mean stretch curvature. Thus

$$J_{i|j} - J_{j|i} = \lambda F\big(I_{i|j} - I_{j|i}\big), \tag{4}$$

where $\lambda = \lambda(x, y)$ is a positive (negative or constant) scalar function on TM. Contracting (4) with y^j implies that

$$J_{i|j} y^j = \lambda F J_i. \tag{5}$$

First, let $\lambda = \lambda(x, y)$ be a non-negative scalar function on TM. Let us put

$$\varphi := J^m J_m.$$

Then, we have

$$\varphi' := \varphi_{|s} y^s = 2J^m J_{m|s} y^s = 2\lambda F \varphi. \tag{6}$$

By definition, F and φ have positive values. If λ is negative (positive), then φ' is negative (positive). By Theorem 3.1, we get

$$\varphi' := \varphi_{|s} y^s = \xi(\varphi) = \overline{\operatorname{Div}}(\varphi\xi).$$

Note that $\xi := y^i\delta/\delta x^i$ is a geodesic vector field on unit sphere tangent bundle SM and $\overline{\mathrm{Div}}(\xi) = 0$. Since M is compact, then SM is compact, too. The volume form ω_{SM} on SM is obtained from volume form ω on M. By Theorem 3.2, we get

$$\int_{SM} \varphi'\omega_{SM} = 0.$$

Since φ' is homogeneous function and its sign is negative (positive), then $\varphi' = 0$. By (6), we have $\varphi = 0$ or $\lambda = 0$. If $\varphi = 0$, then $\mathbf{J} = 0$. If $\lambda = 0$, then $\bar{\Sigma} = 0$. In this case,

$$\mathbf{J}' = J_{i|s}y^s = 0.$$

Thus

$$\mathbf{J}(t) = \mathbf{J}(0),$$

which implies that

$$\mathbf{J}(t) = \mathbf{J}(0)t + \mathbf{I}(t). \tag{7}$$

Letting $t \to \pm\infty$ and using $\|\mathbf{I}\| < \infty$, we get $\mathbf{J}(0) = 0$. Thus $\mathbf{J}(t) = 0$ and F is a weakly Landsberg metric.

Now, suppose that $\lambda =$ constant. In this case, by (5), we get

$$\mathbf{J}(t) = \mathrm{e}^{t\lambda}\,\mathbf{J}(0).$$

Using $\|\mathbf{I}\| < \infty$ and letting $t \to \pm\infty$ implies that

$$\mathbf{J}(0) = 0.$$

Thus $\mathbf{J}(t) = 0$ and F is a weakly Landsberg metric. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we remark that the Berwald frame was founded by Berwald in order to study Finsler surfaces [**3**]. It works under the assumption that the fundamental tensor $g_{ij}(x, y)$ is positive-definite. Then he defined a local field of orthonormal frame (ℓ, m) which is called the Berwald frame.

Proof of Theorem 1.2. Let (M, F) be a Finsler surface. We refer to the Berwald's frame (ℓ^i, m^i), where

$$\ell^i := \frac{y^i}{F},$$

m^i is the unit vector with $\ell_i m^i = 0$, $\ell_i = g_{ij}\ell^j$,and g_{ij} is the fundamental tensor of Finsler metric F. Then the Berwald curvature is given by

$$B^i{}_{jkl} = \frac{1}{F}(I_{,2}m^i - 2I_{,1}\ell^i)m_j m_k m_l,$$

where $I = I(x, y)$ is 0-homogeneous function called the main scalar of F and

$$I_2 = I_{,2} + I_{,1|2}.$$

For more details, see [**17**]. By the above relation, we have

$$B^i{}_{jkl} = -\frac{2y^i}{3F^2}\big(m_j h_{kl} + m_k h_{jl} + m_l h_{jk}\big)I_{,1} + \frac{I_2}{3F}\big(h^i{}_j h_{kl} + h^i{}_k h_{jl} + h^i{}_l h_{jk}\big), \tag{8}$$

where $h_{ij} := m_i m_j$. Therefore, every two-dimensional Finsler metric satisfies

$$B^i{}_{jkl} = (\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})y^i + \lambda(h^i{}_j h_{kl} + h^i{}_k h_{jl} + h^i{}_l h_{jk}), \tag{9}$$

where

$$\mu_i = -\frac{2}{3F^2}I_{,1}m_i \qquad \text{and} \qquad \lambda = \frac{I_2}{3F}.$$

Multiplying (9) with y^j, and using

$$y^j B^i{}_{jkl} = 0, \qquad \text{and} \qquad y^j h^i{}_j = y^j(\delta^i_j - F^{-2}y^i y_j) = 0,$$

imply that

$$y^i\mu_i = 0.$$

Thus by contracting (9) with y_i, we have

$$y_i B^i{}_{jkl} = F^2(\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk}) + \lambda y_i(h^i{}_j h_{kl} + h^i{}_k h_{jl} + h^i{}_l h_{jk}). \tag{10}$$

By using $y_i B^i{}_{jkl} = -2L_{jkl}$ and $y_i h^i{}_m = 0$, the equation (10) reduces to

$$L_{jkl} = -\frac{1}{2}F^2\big(\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk}\big). \tag{11}$$

Contracting (11) with g^{kl} yields

$$J_j = -\frac{3}{2}F^2\mu_j. \tag{12}$$

By contracting i and l in (9), we get

$$2E_{ij} = 3\lambda h_{ij}. \tag{13}$$

Plugging (12) and (13) in (9), yields

$$B^i{}_{jkl} = \frac{2}{3}\big(E_{kl}h^i_j + E_{jl}h^i_k + E_{jk}h^i_l\big) - \frac{2}{3F^2}\big(J_j h_{kl} + J_k h_{jl} + J_l h_{jk}\big)y^i. \tag{14}$$

Taking a horizontal derivation of (14) implies that

$$\begin{aligned} B^i{}_{jkl|s} = {} & \frac{2}{3}\big(E_{kl|s}h^i_j + E_{jl|s}h^i_k + E_{jk|s}h^i_l\big) - \frac{2}{3F^2}\big(J_{j|s}h_{kl} + J_{k|s}h_{jl} + J_{l|s}h_{jk}\big)y^i \\ & + \frac{4}{3F^2}\big(J_j L_{kls} + J_k L_{jls} + J_l L_{jks}\big)y^i. \end{aligned} \tag{15}$$

Contracting (15) with y^s gives us

$$B^i{}_{jkl|s}y^s = \frac{2}{3}\big(H_{kl}h^i_j + H_{jl}h^i_k + H_{jk}h^i_l\big) - \frac{2}{3F^2}\big(J'_j h_{kl} + J'_k h_{jl} + J'_l h_{jk}\big)y^i, \tag{16}$$

where $J'_i := J_{i|s}y^s$ is the horizontal derivation of mean Landsberg tensor along Finslerian geodesics. By assumption, F is weakly stretch metric, and then (16) reduces to

$$B^i{}_{jkl|s}y^s = \frac{2}{3}\big(H_{kl}h^i_j + H_{jl}h^i_k + H_{jk}h^i_l\big). \tag{17}$$

According to (17), if $\mathbf{H} = 0$, then $\widetilde{\mathbf{B}} = 0$. The converse is trivial. $\square$

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