CRITICAL POINT EQUATION
ON A COMPACT \((k,\mu)\)-ALMOST CO-KÄHLER MANIFOLD

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ABSTRACT. Our aim is to study critical point equation conjecture on a compact \((k,\mu)\)-almost co-Kähler manifold. We prove that if a compact \((k,\mu)\)-almost co-Kähler manifold of dimension greater than three satisfies critical point equation, then either \(\mu\) is constant or the manifold is an Einstein manifold provided \(k < 0\).

1. INTRODUCTION

Let \((M, g)\) be a compact (without boundary) oriented Riemannian manifold with dimension \((2n + 1) \geq 3\). Also, assume that \(\mathcal{N}\) is the set of Riemannian metrics on \(M\) with unit volume and \(\mathcal{T} \subset \mathcal{N}\) is the subset of metrics with constant scalar curvature. Now, the total scalar curvature functional \(\mathcal{R}: \mathcal{N} \to \mathbb{R}\) is defined as follows:

\[
\mathcal{R}(g) = \int_M r_g dM_g,
\]

where \(r_g\) and \(dM_g\) are the scalar curvature and the volume form determined by the metric, respectively.

If we consider the above functional restricted to \(\mathcal{T}\), then the Euler-Lagrangian equation is given by

\[
\left(\frac{r}{2n} - S\right)\lambda - \text{Hess} \lambda = S - \frac{r}{2n + 1} g,
\]

for some smooth function \(\lambda\) on \(M\), where \(S\), \(r\), and \(\text{Hess} \lambda\), respectively, the Ricci tensor, scalar curvature, and Hessian of the smooth function \(\lambda\).

In particular, if \(\lambda\) is constant, then \(\lambda = 0\) and the metric will be Einstein. Therefore, we have the following definition.

**Definition 1.1.** A compact Riemannian manifold \((M, g)\) of dimension \((2n + 1) \geq 3\) with constant scalar curvature and unit volume together with a non-constant smooth potential function \(\lambda\) satisfying (1.2), is called critical point equation metric.

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The conjecture “a CPE metric will be Einstein” was proposed in 1984 by A. Besse [1], but has yet to be proved in different manifolds by many authors. Lafontaine [11] proved that the conjecture is true for a locally conformally flat manifold. Hwang [9] proved the CPE conjecture provided $\lambda \geq -1$. In [3], Chang et. al. were able to solve the conjecture for a manifold satisfying the parallel Ricci tensor condition, also in [22], they proved that if the manifold with the critical point metric has harmonic curvature, then it is isometric to a standard sphere. Several authors ([10], [12], [16], [17]) and many others have studied CPE metric in compact Riemannian and contact metric manifolds.

Motivated by the above studies in the present paper, we consider CPE conjecture on $(k,\mu)$-almost co-Kähler manifolds. The present paper is organized as follows: after introduction in Section 2, we discuss some preliminaries of almost co-Kähler manifolds. In Section 3, we recall some fundamental formulas and properties of $(k,\mu)$-almost co-Kähler manifolds. Section 4 is devoted to prove our main result. Our main Theorem can be presented as follows:

**Theorem 1.2.** If $M^{2n+1}(\phi,\xi,\eta,g)$, $n > 1$, be a compact $(k,\mu)$-almost co-Kähler manifold with $k < 0$ and satisfies critical point equation, then either $\mu$ is constant or the manifold is an Einstein manifold.

### 2. Preliminaries

Let $M$ be a $(2n + 1)$-dimensional smooth manifold and if there exist a $(1,1)$-type tensor field $\phi$, a vector field $\xi$, and a 1-form $\eta$ such that

\begin{equation}
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0,
\end{equation}

then we say that the triplet $(\phi,\xi,\eta)$ is an almost contact structure on $M^{2n+1}$. In general, a smooth manifold $M^{2n+1}$ equipped with an almost contact structure is called an almost contact manifold. If on an almost contact manifold $M$, there exists a Riemannian metric $g$ satisfying

\begin{equation}
g(\phi U,\phi V) = g(U,V) - \eta(U)\eta(V)
\end{equation}

for any vector fields $U$, $V$, then $M$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\phi,\xi,\eta,g)$. From (2.1), it follows that

\begin{equation}
g(U,\phi V) = -g(\phi U,V), \quad g(U,\xi) = \eta(U),
\end{equation}

for all vector fields $U$, $V$. An almost contact metric structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

\begin{equation}
J(U,f \frac{d}{dt}) = (\phi U - f\xi,\eta(U)\frac{d}{dt})
\end{equation}

is integrable, where $U$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$, and $f$ is a smooth function on $M \times \mathbb{R}$. An almost contact metric structure becomes a contact metric
structure if
\[(2.4)\quad g(U, φV) = dη(U, V) = Φ(U, V)\]
for all \(U, V\) tangent to \(M\).

A normal almost contact metric manifold is said to be Sasakian. That is, an almost contact metric manifold is Sasakian if and only if
\[(2.5)\quad (\nabla_U φ)V = g(X, V)ξ - η(V)U\]
for any vector fields \(U, V\).

By an almost co-Kähler manifold we mean an almost contact metric manifold such that both the 1-form \(η\) and the 2-form \(Φ\) are closed. In particular, an almost co-Kähler manifold is said to be a co-Kähler manifold if the associated almost contact structure is normal, which is also equivalent to \(\nabla φ = 0\) or to \(\nabla Φ = 0\). It is well known that the Riemannian product of a real line and a (almost) Kähler manifold admit a (almost) co-Kähler structure. However, there exist some examples of (almost) co-Kähler manifolds which are not globally the product of a (almost) Kähler manifold and a real line (see, for example, Chinea et al. \[2\], Marrero and Padrón \[13\]). Note that almost co-Kähler manifolds are just the almost cosymplectic manifolds studied by Wang \([19\], \[20\]).

On an almost co-Kähler manifold \((M^{2n+1}, φ, ξ, η, g)\), we set \(h = \frac{1}{2} ξ φ\) and \(h' = h ◦ φ\) (both \(h\) and \(h'\) are symmetric operators with respect to the metric \(g\)). The following formulas can be found in Dacko \[5\], Endo \([7\], \[8\]) and Olszak \([14\], \[15\]):
\[(2.6)\quad hξ = 0, \quad hφ + φh = 0, \quad \text{tr} h = \text{tr} h',\]
\[(2.7)\quad \nabla ξ φ = 0, \quad \nabla ξ = h', \quad \text{div} ξ = 0,\]
\[(2.8)\quad S(ξ, ξ) + ∥h∥^2 = 0.\]
Here 'tr' and 'div' denote the trace and divergence operators, respectively, with respect to the metric \(g\). The Ricci tensor \(S\) is defined by \(S(U, V) = \text{tr} [\cdot → R(\cdot, U)V]\) and \(Q\) is the Ricci operator defined by \(g(QU, V) = S(U, V)\).

If we put \(l = R(\cdot, ξ)ξ\), then we can also show
\[(2.9)\quad φlφ - l = 2h^2,\]
where the Riemannian curvature tensor \(R\) is defined by
\[R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W.\]
The \((k, µ)\)-nullity distribution \(N(k, µ)\) on a contact metric manifold is defined by
\[N(k, µ) : p → N_p(k, µ) = \{W ∈ T_pM : R(U, V)W = (kI + µh)(g(V, W)U - g(U, W)V)\}\]
for all \(U, V ∈ T_pM\), where \((k, µ) ∈ \mathbb{R}^2\). A contact metric manifold \(M^{2n+1}\) with \(ξ ∈ N(k, µ)\) is called a \((k, µ)\)-manifold.
3. \((k, \mu)\)-almost co-Kähler manifolds

By a \((k, \mu)\)-almost co-Kähler manifold, we mean an almost co-Kähler manifold such that the Reeb vector field \(\xi\) belongs to the generalized \((k, \mu)\)-nullity distribution, that is,

\[ R(U, V)\xi = k[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV] \]

for all \(U, V \in M^{2n+1}\), where \(k, \mu\) are smooth functions on \(M^{2n+1}\).

In this paper, we consider a \((k, \mu)\)-almost co-Kähler manifold as a \((k, \mu)\)-almost co-Kähler manifold with \(k < 0\) (for more details, see [8]). Such manifolds were generalized to \((k, \mu, v)\)-spaces by Dacko and Olszak [5]. Recently in [18], Suh and De studied \((k, \mu)\)-almost co-Kähler manifold. From equation (3.1), we obtain

\[ l = -k\phi^2 + \mu k, \]

which together with (2.9) gives

\[ h^2 = k\phi^2. \tag{3.2} \]

Now, we state the following Lemma, given by Wang [19].

**Lemma 3.1.** Let \(M^{2n+1}(\phi, \xi, \eta, g)\) be a \((k, \mu)\)-almost co-Kähler manifold of dimension greater than 3 with \(k < 0\). Then the Ricci operator \(Q\) of \(M^{2n+1}\) is given by

\[ QV = \mu hV + 2nk\eta(V)\xi \]

for all \(U \in \chi(M)\), where \(D\) denotes the gradient operator with respect to \(g\).

Moreover, the scalar curvature of \(M^{2n+1}\) is \(2nk\).

4. Proof of the main Theorem

In this section, we consider a compact \((k, \mu)\)-almost co-Kähler manifold satisfying the critical point equation.

Taking trace of the equation (1.2), we have

\[ \Delta \lambda = -\frac{r\lambda}{2n}. \tag{4.1} \]

Using (4.1) in (1.1), we obtain

\[ \nabla_U D\lambda = (\lambda + 1)QU + fU, \quad \text{where } f = -r\left(\lambda \frac{1}{2n} + \frac{1}{2n+1}\right) \]

for any \(U \in \chi(M)\), where \(D\) denotes the gradient operator with respect to \(g\). Taking the covariant derivative of (4.2) with respect to \(V\), we get

\[ \nabla_V \nabla_U D\lambda = (V\lambda)QU + (\lambda + 1)(\nabla_V Q)U + (Vf)U + j\nabla_V U \]

for any \(U, V \in \chi(M)\). Similarly, we get

\[ \nabla_U \nabla_V D\lambda = (U\lambda)QV + (\lambda + 1)(\nabla_U Q)V + (Uf)V + j\nabla_V V. \]

Also, we have

\[ \nabla_{[U,V]} D\lambda = (\lambda + 1)Q[U,V] + f[U,V], \]

where \([U,V]\) denotes the Lie bracket of \(U\) and \(V\).
and using (4.3), (4.4), and (4.5), we have

\[
R(U,V)D\lambda = \nabla_U \nabla_V D\lambda - \nabla_V \nabla_U D\lambda - \nabla_{[U,V]} D\lambda \\
= (U\lambda)QV - (V\lambda)QU + (\lambda + 1)[(\nabla_U Q)V - (\nabla_V Q)U] \\
+ (Uf)V - (Vf)U.
\]

(4.6)

By setting \(U = \xi\) in (4.6) and using (3.3), we have

\[
R(\xi,V)D\lambda = \mu(\xi\lambda)hV + 2nk(\xi\lambda)\eta(V)\xi - 2nk(V\lambda)\xi \\
+ (\lambda + 1)\left\{ (\nabla_\xi Q)V - (\nabla_V Q)\xi \right\} + (jf)V - (Vf)\xi.
\]

(4.7)

Taking covariant derivative of (3.3) with respect to arbitrary vector field \(U\), we get

\[
(\nabla_U Q)V = (U\mu)hV + \mu(\nabla_U h)V + 2nk(\nabla_U \eta)(V)\xi + 2nk\eta(V)\nabla_U \xi.
\]

(4.8)

In view of (4.8) and using (2.7), we have

\[
(\nabla_\xi Q)V - (\nabla_V Q)\xi = (\xi\mu)hV + \mu\left\{ (\nabla_\xi h)V - (\nabla_V h)\xi \right\} - 2nk\phi V.
\]

(4.9)

Taking inner product of both sides of (4.7) with respect to \(\xi\) and using the above relation yield

\[
g(R(\xi,V)D\lambda,\xi) = 2nk(\xi\lambda)\eta(V) - 2nk(V\lambda) + (jf)\eta(V) - (Vf).
\]

(4.10)

On the other hand, from (3.1), we get

\[
g(R(\xi,V)\xi,D\lambda) = k[\eta(V)g(D\lambda,\xi) - g(V,D\lambda)] - \mu g(hV,D\lambda).
\]

(4.11)

Combining (4.10) and (4.11), we have

\[
2nk(\xi\lambda)\eta(V) - 2nk(V\lambda) + (jf)\eta(V) - (Vf) \\
= k[\eta(V,D\lambda) - \eta(V)g(D\lambda,\xi)] + \mu g(hV,D\lambda).
\]

Removing \(V\) from the above equation, we get

\[
2nk(\xi\lambda)\xi + (jf)\xi = (2n + 1)kD\lambda - k\eta(D\lambda)\xi + Df + \mu hD\lambda.
\]

(4.12)

In virtue of \(f = -r(\frac{\lambda}{2n} + \frac{1}{m+1})\) and Lemma 3.1, we obtain

\[
\xi f = -k(\xi\lambda) \quad \text{and} \quad Df = -k(D\lambda).
\]

(4.13)

With the help of the above relations, (4.12) reduces to

\[
2nk(\xi\lambda)\xi - D\lambda = \mu hD\lambda.
\]

(4.14)

Applying \(h\) on both sides of (4.14) and since \(k < 0\), using (3.2), we obtain

\[
2nhD\lambda = \mu(D\lambda - \eta(D\lambda)\xi).
\]

Using the above expression in (4.14), we get

\[
(4n^2k + \mu^2)(\xi\lambda)\xi - D\lambda = 0.
\]

(4.15)

Now, there arise two cases:

**Case 1.** Let \(\mu^2 = -4n^2k\), this implies \(\mu\) is constant.

**Case 2.** Let \(D\lambda = (\xi\lambda)\xi\), and taking covariant derivative of this equation with
respect to $U$ and using (2.7), we obtain $\nabla_U D\lambda = U(\xi(\lambda))\xi + (\xi)h'U$. Combining this relation with (4.2) yields
\[
(\lambda + 1)QU = U(\xi(\lambda))\xi + (\xi)h'U - fU.
\]
Equating the above relation with (3.3), we infer that
\[
(\lambda + 1)(\mu hU + 2nk\eta(U)\xi) = U(\xi(\lambda))\xi + (\xi)h'U - fU. \tag{4.15}
\]
Taking inner product of the above relation with $V$, we get
\[
(\lambda + 1)(\mu g(hU,V) + 2nk\eta(U)\eta(V)) = U(\xi(\lambda))\eta(V) + (\xi)g(h'U,V) - f g(U,V). \tag{4.16}
\]
Contracing (4.16), we obtain
\[
\xi(\xi(\lambda)) = 2nk(\lambda + 1) + (2n + 1)f. \tag{4.17}
\]
Since $\nabla_\xi D\lambda = \xi(\xi(\lambda))\xi$, putting $U = \xi$ in (4.2), we deduce that
\[
\xi(\xi(\lambda)) = \{2nk(\lambda + 1) + f\}. \tag{4.18}
\]
Now, (4.17) and (4.18) together imply $f = 0$, then from (4.2), we get $\lambda = -\frac{2n}{2n+1}$.
This implies that the manifold is an Einstein manifold.
This completes the proof.

Since an almost cosymplectic manifold and an almost co-Kähler manifold are
the same, we can say the following:
If we consider $k = -1$ and use $\mu^2 = -4n^2k$, we have $|\mu| > 2$, then the compact
$(k,\mu)$-almost co-Kähler manifold $M$ is locally isometric to a Lie group (see [6, Proposition 5.3]). Thus we state the following corollary.

**Corollary 4.1.** If a compact $(k,\mu)$-almost co-Kähler manifold of dimension
greater than 3 satisfies critical point equation, then either the manifold is locally
isometric to a Lie group or the manifold is an Einstein manifold provided $k = -1$.

**Remark.** If we consider $k=0$ in compact $(k,\mu)$-almost co-Kähler manifold,
then in view of the Theorem 2 of [4], we have $\xi$ is killing and the manifold $M$ is
locally isometric to a product of a real line and an almost Kähler manifold.

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