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ON CURVES OF CONSTANT BREADTH IN A 3-DIMENSIONAL LIE GROUP

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Abstract. In this paper, we investigate the properties of curves of constant breadth in a 3-dimensional Lie group. Also, we find the condition of general helix as constant breadth curves and construct constant breadth curves which the tangent component of the curve vanishes.

1. Introduction

In 1780, Euler introduced curves of constant breadth and studied those on a Euclidean plane [**6**]. After him, many mathematicians investigated the geometric properties of the plane curves of constant breadth [**4**], [**12**], [**15**]. As a extension, Fujiwara [**7**] defined constant breadth for space curves and he obtained a problem to determine whether there exists space curve of constant breadth or not. Furthermore, Blaschke [**2**] defined the curves of constant breadth on a sphere. Many geometers have been interested in studying curves of constant breadth when an ambient space is the Euclidean space and the Minkowski space ([**8**], [**10**], [**11**], [**14**], [**16**], [**17**], etc).

In this paper, we study curves of constant breadth in a 3-dimensional Lie group and construct them.

2. Preliminaries

Let G be a Lie group with a bi-invariant Riemannian metric $\langle\ ,\ \rangle$ and $\mathcal{G}$ be the Lie algebra of G. Then $\mathcal{G}$ is isomorphic to T_eG, where e is identity of G. Moreover, the following equations

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X, Y] \tag{2.2}$$

are satisfied with respect to bi-invariant metric for all $X, Y, Z \in \mathcal{G}$, where D is the Levi-Civita connection of Lie group G.

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Let $\alpha\colon I\subset R\to G$ be a parameterized curve with parameter t and $\{V_1, V_2, \cdots, V_n\}$ be an orthonormal basis of $\mathcal{G}$. In this case, we write that any smooth vector fields W and Z along the curve α as $W=\sum_{i=1}^{n} w_iV_i$ and $Z=\sum_{i=1}^{n} z_iV_i$, where $w_i\colon I\to R$ and $z_i\colon I\to R$ are smooth functions. Furthermore, the Lie bracket of two vector fields W and Z, is given by

$$[W,Z]=\sum_{i,j=1}^{n} w_iz_j[V_i,V_j]. \tag{2.3}$$

Let $D_{\alpha'(t)}W$ be the covariant derivative of W along the curve α, $V_1=\alpha'$ and $\dot{W}=\sum_{i=1}^{n}\dot{w}_iV_i$, where $\dot{w}_i=\mathrm{d}w_i/\mathrm{d}t$. Then we have (cf. [**13**])

$$D_{\alpha'(t)}W=\dot{W}+\frac{1}{2}[V_1,W]. \tag{2.4}$$

Let $\alpha\colon I\to G$ be a parameterized curve with the Frenet frame $\{T,N,B\}$ in a 3-dimensional Lie group G and s be arc-length of the curve α. Then the Frenet formulas of the curve α is given by

$$D_TT=k_1N,\quad D_TN=-k_1T+k_2B,\quad D_TB=-k_2N,$$

where k_1 and k_2 are the curvature functions of α.
For later use we define smooth function $\overline{k}_2$ as follows:

$$\overline{k}_2(s)=\frac{1}{2}\langle[T,N],B\rangle. \tag{2.5}$$

Proposition 2.1 ([**18**])**.** *Let α be a parameterized curve in a 3-dimensional Lie group G with a bi-invariant metric. Then we have*

$$\begin{aligned}[T,N]&=\langle[T,N],B\rangle B=2\overline{k}_2B,\\ [B,T]&=\langle[B,T],N\rangle N=2\overline{k}_2N,\\ [N,B]&=\langle[N,B],T\rangle T=2\overline{k}_2T.\end{aligned}$$

Remark. Let G be a 3-dimensional Lie group with a bi-invariant metric. Then it is one of the Lie groups $SO(3)$, S^3 or a commutative group, and the following statements hold (see [**5**]):

(i) If G is $SO(3)$, then $\overline{k}_2(s)=\frac{1}{2}$.
(ii) If G is $S^3\cong SU(2)$, then $\overline{k}_2(s)=1$.
(iii) If G is a commutative group, then $\overline{k}_2(s)=0$.

Theorem 2.1 ([**3**])**.** *A parameterized curve in a 3-dimensional Lie group G with a bi-invariant metric is a general helix if and only if*

$$k_2(s)-\overline{k}_2(s)=ck_1(s), \tag{2.6}$$

where c is constant.

From (2.6), a curve with $k_1\neq 0$ is a general helix if and only if $(\frac{k_2-\overline{k}_2}{k_1})(s)$ is constant. In the Euclidean sense if both $k_1(s)\neq 0$ and $k_2(s)-\overline{k}_2(s)$ are constants, it is a cylindrical helix. We call such a curve a *circular helix* in a 3-dimensional Lie group G.

3. Curves of constant breadth in Lie group

In this section, we define space curves of constant breadth in a 3-dimensional Lie group G and construct them.

Definition 3.1. A curve $\alpha\colon I \to G$ in a 3-dimensional Lie group G is called a curve of constant breadth if there exists a curve $\beta\colon I \to G$ such that at the corresponding points of curves, the parallel tangent vectors of α and β at $\alpha(s)$ and $\beta(s^*)$ at $s, s^* \in I$ are opposite directions and the distance between these points is always constant. In this case, (α, β) is called a curve pair of constant breadth.

Let now (α, β) be a curve pair of constant breadth and s, s^* be arc-length of α and β, respectively. Then we may write the following equation

$$\beta(s^*) = \alpha(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s), \tag{3.1}$$

where m_i $(i = 1, 2, 3)$ are smooth functions of s.
Differentiating (3.1) with respect to s and using (2.4), we obtain

$$\begin{aligned} T^*\frac{\mathrm{d}s^*}{\mathrm{d}s} &= (1 + m_1' - m_2k_1)T + (m_2' + m_1k_1 - m_3k_2 + m_3\overline{k}_2)N \\ &\quad + (m_3' + m_2k_2 - m_2\overline{k}_2)B, \end{aligned} \tag{3.2}$$

where T^* denotes the tangent vector of β. Here the prime $'$ denotes the derivative with respect to s. Since $T = -T^*$, from (3.2), we have

$$1 + m_1' - m_2k_1 = -\frac{\mathrm{d}s^*}{\mathrm{d}s}, \tag{3.3a}$$

$$m_2' + m_1k_1 - m_3(k_2 - \overline{k}_2) = 0, \tag{3.3b}$$

$$m_3' + m_2(k_2 - \overline{k}_2) = 0. \tag{3.3c}$$

If φ is the angle between the tangent vector of α and a given fixed direction, the curvature of α is $\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}s}$. We put $\rho = 1/k_1, \rho^* = 1/{k_1}^*$, where ${k_1}^*$ is the curvature function of β. If we take $f(\varphi) = \rho + \rho^*$, then equation (3.3) can be rewritten as

$$\frac{\mathrm{d}m_1}{\mathrm{d}\varphi} = m_2 - f(\varphi), \tag{3.4a}$$

$$\frac{\mathrm{d}m_2}{\mathrm{d}\varphi} = -m_1 + m_3\rho(k_2 - \overline{k}_2), \tag{3.4b}$$

$$\frac{\mathrm{d}m_3}{\mathrm{d}\varphi} = -m_2\rho(k_2 - \overline{k}_2). \tag{3.4c}$$

Differentiating (3.4b) with respect to φ, we obtain the following equation

$$\begin{aligned} \rho(k_2 - \overline{k}_2)(m_2 - f(\varphi)) = &-\rho(k_2 - \overline{k}_2)\frac{\mathrm{d}^2m_2}{\mathrm{d}\varphi^2} - m_2\rho^3(k_2 - \overline{k}_2)^3 \\ &+ (m_1 + \frac{\mathrm{d}m_2}{\mathrm{d}\varphi})\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\rho(k_2 - \overline{k}_2)\right). \end{aligned} \tag{3.5}$$

If the distance between the opposite points of α and β is constant, then

$$\|\alpha - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant},$$

which implies

$$m_1\frac{\mathrm{d}m_1}{\mathrm{d}\varphi}+m_2\frac{\mathrm{d}m_2}{\mathrm{d}\varphi}+m_3\frac{\mathrm{d}m_3}{\mathrm{d}\varphi}=0. \tag{3.6}$$

Combining (3.4) and (3.6), we get

$$m_1\left(m_2-\frac{\mathrm{d}m_1}{\mathrm{d}\varphi}\right)=0. \tag{3.7}$$

We consider two cases separately.

<u>*Case 1.*</u> Suppose $m_2=\frac{\mathrm{d}m_1}{\mathrm{d}\varphi}$. Then $f(\varphi)=0$ in (3.4a), it follows that (3.5) becomes

$$\begin{aligned} m_2\rho(k_2-\overline{k}_2)+\rho(k_2-\overline{k}_2)\frac{d^2m_2}{\mathrm{d}\varphi^2}+m_2\rho^3(k_2-\overline{k}_2)^3 \\ -(m_1+\frac{\mathrm{d}m_2}{\mathrm{d}\varphi})\frac{\mathrm{d}}{\mathrm{d}\varphi}\big(\rho(k_2-\overline{k}_2)\big)=0. \end{aligned} \tag{3.8}$$

We consider m_1 is non-zero constant and $m_2=0$. Then from (3.8), $\rho(k_2-\overline{k}_2)=\frac{k_2-\overline{k}_2}{k_1}=$ constant. It shows that α is a general helix.

<u>*Case 2*</u>. Suppose $m_1=0$. Then equation (3.4) can be rewritten in the form

$$m_2=f(\varphi), \tag{3.9a}$$

$$\frac{\mathrm{d}m_2}{\mathrm{d}\varphi}=m_3\rho(k_2-\overline{k}_2), \tag{3.9b}$$

$$\frac{\mathrm{d}m_3}{\mathrm{d}\varphi}=-m_2\rho(k_2-\overline{k}_2). \tag{3.9c}$$

Differentiating (3.9b) with respect to φ, we have

$$\frac{\mathrm{d}^2m_2}{\mathrm{d}\varphi^2}-\frac{\frac{\mathrm{d}}{\mathrm{d}\varphi}(\rho(k_2-\overline{k}_2))}{\rho(k_2-\overline{k}_2)}\frac{\mathrm{d}m_2}{\mathrm{d}\varphi}+\big(\rho(k_2-\overline{k}_2)\big)^2m_2=0. \tag{3.10}$$

To solve this equation, let

$$R(\varphi)=\frac{\frac{\mathrm{d}}{\mathrm{d}\varphi}\big(\rho(k_2-\overline{k}_2)\big)}{\rho(k_2-\overline{k}_2)},\qquad S(\varphi)=\big(\rho(k_2-\overline{k}_2)\big)^2 \tag{3.11}$$

and let the transformation be $z=\theta(\varphi)$ with $\frac{\mathrm{d}z}{\mathrm{d}\varphi}=(S(\varphi)/a^2)^{1/2}$, where a is any positive integer. Then, equation (3.10) may be written in the form

$$\frac{\mathrm{d}^2m_2}{\mathrm{d}z^2}+\left(\frac{\frac{\mathrm{d}^2z}{\mathrm{d}\varphi^2}-R(\varphi)\frac{\mathrm{d}z}{\mathrm{d}\varphi}}{(\frac{\mathrm{d}z}{\mathrm{d}\varphi})^2}\right)\frac{\mathrm{d}m_2}{\mathrm{d}z}+\frac{S(\varphi)}{(\frac{\mathrm{d}z}{\mathrm{d}\varphi})^2}m_2=0. \tag{3.12}$$

Since $\frac{\mathrm{d}z}{\mathrm{d}\varphi}=(S(\varphi)/a^2)^{1/2}$, equation (3.12) becomes

$$\frac{\mathrm{d}^2m_2}{\mathrm{d}z^2}+a^2m_2=0, \tag{3.13}$$

and its general solution is given by

$$m_2=c\cos(az+b)=c\cos\left(\int_0^{\varphi}\frac{k_2-\overline{k}_2}{k_1}\mathrm{d}\varphi+b\right), \tag{3.14}$$

which implies that from (3.9b), we have

$$m_3 = -c\sin\left(\int_0^{\varphi} \frac{k_2 - \overline{k}_2}{k_1}\mathrm{d}\varphi + b\right), \tag{3.15}$$

where b and c are constants of integration.
Using (3.14) and (3.15), an infinite number of β can be derived and the distance between the corresponding points of a pair curve of constant breadth is c.

Thus, we have the following theorems

Theorem 3.1. *The curve of constant breadth with the tangent component $m_1 =$ non-zero constant and the principal normal component $m_2 = 0$ is a general helix in a 3-dimensional Lie group.*

Example 3.2. We consider a curve parameterized by

$$\alpha(s) = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}s\right).$$

The curve has the curvature functions $k_1 = \frac{1}{\sqrt{2}}$ and $k_2 = \frac{1}{\sqrt{2}}$. We put $m_1 = 1, m_2 = 0$, $m_3 = -1$ and $\overline{k}_2 = 0$. Then we can construct the curve of constant breadth of the curve α, this is, it is given by

$$\beta(s) = \left(\frac{1}{\sqrt{2}}\cos s - \sqrt{2}\sin s, \frac{1}{\sqrt{2}}\sin s + \sqrt{2}\cos s, \frac{1}{\sqrt{2}}s\right).$$

By a long computation, the curvature functions of the curve β are given by $(k_1)_\beta = \frac{\sqrt{10}}{6}$ and $(k_2)_\beta = \frac{\sqrt{2}}{6}$. Thus, the curve β is a helix in a commutative group G.

Theorem 3.3. *Let (α, β) be a pair curve of constant breadth in a 3-dimensional Lie group. If α is a curve with $m_1 = 0$, then a curve β is expressed as*

$$\beta = \alpha + c\cos\left(\int_0^{\varphi} \frac{k_2 - \overline{k}_2}{k_1}\mathrm{d}\varphi + b\right)N - c\sin\left(\int_0^{\varphi} \frac{k_2 - \overline{k}_2}{k_1}\mathrm{d}\varphi + b\right)B.$$

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