Acta Math. Univ. Comenianae
Vol. XC, 3 (2021), pp. 245–258

SIGNLESS LAPLACIAN SPECTRAL DETERMINATION OF PATH-FRIENDSHIP GRAPHS

R. SHARAFDINI and A. Z. ABDIAN

Abstract. A graph G is said to be DQS if there is no other non-isomorphic graph with the same signless Laplacian spectrum as G. Let k, t_i $(1 \leq i \leq k)$, and s be natural numbers. A path-friendship graph, $G_{s,t_1,\ldots,t_k}$, is a graph of order $n = 2s + t_1 + \cdots + t_k + 1$ which consists of s triangles and k paths of lengths $t_1, t_2, \ldots, t_k$ sharing a common vertex. In this paper, we show that these graphs are DQS and using this result, we respond to a conjecture in [F. Wen, Q. Huang, X. Huang and F. Liu, *The spectral characterization of wind-wheel graphs*, Indian J. Pure Appl. Math. **46**(2015), 613–631].

1. Introduction

In the past decades, graphs that are determined by their spectrum have received more and more attention since they have been applied to several fields such as randomized algorithms, combinatorial optimization problems, and machine learning. An important part of spectral graph theory is devoted to determining whether given graphs or classes of graphs are determined by their spectra or not. So, finding and introducing any class of graphs which are determined by their spectra can be an interesting and important problem. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The line graph of G is denoted by $\mathcal{L}(G)$. We denote the degree sequence of G by $\deg(G) = (d_1, d_2, \ldots, d_n)$, where $d_i = d_i(G)$ is the i-th largest vertex degree of G for $i = 1, \ldots, n$.

We denote the adjacency matrix of G by $A(G)$. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G, where $D(G)$ denotes the degree matrix of G; namely $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$.

The characteristic polynomials of G with respect to $A(G)$, and $Q(G)$, respectively, are denoted by $\varphi(G, \lambda) = \det(\lambda I - A(G))$ and $\psi(G, q) = \det(qI - Q(G))$. Conventionally, the adjacency eigenvalues and signless Laplacian eigenvalues of graph G are, respectively, ordered in non-increasing sequence as follows: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $q_1 \geq q_2 \geq \cdots \geq q_n$. The multi-set $\operatorname{Spec}_Q(G) = \{[q_1]^{m_1}, [q_2]^{m_2}, \ldots, [q_n]^{m_n}\}$

Received February 20, 2020; revised March 5, 2021.
2020 *Mathematics Subject Classification.* Primary 05C50.
Key words and phrases. Path-friendship graph; DQS graph; Q-cospectral.

of eigenvalues of $Q(G)$ is called the signless Laplacian spectrum (Q-spectrum) of G, where m_i denote the multiplicities of q_i. Two graphs are Q-cospectral (A-cospectral) if they have the same Q-spectrum (A-spectrum). A graph G is said to be DQS if there is no other non-isomorphic graph Q-cospectral with it.

By sG, we mean s copies of a graph G, where s is a natural number. In fact, $sG = \underbrace{G \cup \cdots \cup G}_{s \text{ times}}$. Let G and H be two graphs with specific vertices $v \in V(G)$ and $u \in V(H)$. A coalescence $G \circ H(u, v)$ is the graph obtained from graphs G and H by identifying u and v in $G \cup H$. Let F_s denote the friendship graph consisting of s triangles intersecting in a single vertex. Note that F_s may be described as a coalescence of s cycles C_3. A wind-wheel graph $G_{s,t}$ on $2s + t + 1$ vertices is the graph obtained by appending s triangle(s) to a pendant vertex of the path P_{t+1}. The lollipop graph of order n, denoted by $H_{n,p}$, is obtained by appending a cycle C_p to a pendant vertex of a path P_{n-p}. By $B_{r,s}$, we denote a butterfly graph that consists of s triangles sharing a common vertex having an additional r pendant vertices. A tree with exactly one vertex v of degree greater that 2 is called a starlike tree. By $T_{t_1,t_2,\ldots,t_k}$, we denote the starlike tree with maximum degree k such that

$$T_{t_1,t_2,\ldots,t_k} - v = P_{t_1} \cup P_{t_2} \cup \cdots \cup P_{t_k},$$

where v is the vertex of degree k and $t_1, t_2, \ldots, t_k$ are any positive integers. We may describe a starlike tree as a coalescence of $P_{t_1+1}, P_{t_2+1}, \ldots, P_{t_k+1}$. In fact, if v_i is a specific pendant vertex of P_{t_i+1} for $i = 1, \ldots, k$, then we have

$$T_{t_1,t_2,\ldots,t_k} = P_{t_1+1} \circ P_{t_2+1} \circ \cdots \circ P_{t_k+1}(v_1, v_2, \ldots, v_k).$$

A path-friendship graph, $G_{s,t_1,\ldots,t_k}$, is a graph of order $n = 2s + t_1 + \cdots + t_k + 1$, which consists of s triangles and k paths of lengths $t_1, t_2, \ldots, t_k$ sharing a common vertex. Note that $G_{s,t_1,\ldots,t_k} = F \circ T(u, v)$, where u and v are, respectively, the unique vertices of $F = F_s$ and $T = T_{t_1,t_2,\ldots,t_k}$ with the maximum degrees $2s$ and k, respectively. It is worth noting that $G_{s,\underbrace{1,\ldots,1}_{r \text{ times}}} = B_{r,s}$ and for any positive integer k, $G_{s,\underbrace{0,\ldots,0}_{k \text{ times}}} = F_s$. A rose graph with p petals (or p-rose graph) is a graph obtained by taking p cycles with just a vertex in common.

Van Dam and Haemers [**12**] conjectured that almost all graphs are determined by their spectra. Nevertheless, the set of graphs that are known to be determined by their spectra is too small. So, discovering infinite classes of graphs that are determined by their spectra can be an interesting problem. About the background of the question "Which graphs are determined by their spectrum?", we refer to [**12**]. We are interested in DQS graphs being a coalescence of DQS graphs. All paths, starlike trees except $K_{1,3}$, friendship graphs, butterfly graph, and wind-wheel graphs and roses (for '$p \geq 3$', are DQS, (see [**1, 13, 2, 9, 12, 15**]). In [**15**], it was asked whether path-friendship graphs are determined by their Q-spectrum. In this paper, we prove that all path-friendship graphs are DQS.

2. Some definitions and preliminaries

In this section, some useful established results which play an important role throughout this paper, are presented.

Lemma 2.1 ([**6**]). *Let G be a graph with second maximum degree $d_2(G)$. Then $q_2(G) \geq d_2(G)-1$. If the equalsity holds, then the maximum and second maximum degree vertices are adjacent and $d_1(G) = d_2(G)$. Moreover, if G is connected, then $q_n(G) < d_n(G)$.*

Lemma 2.2 ([**3, 4**]). *Let G be a connected graph of order $n \geq 2$. Then*

(i) *$q_1(G) \leq \max\{d(v) + m(v) \mid v \in V(G)\}$, where $m(v) = \sum \frac{d(u)}{d(v)}$ and the sum is extended to the neighborhood of v.*

(ii) *$q_1(G) \geq d_1(G) + 1$ with equalsity if and only if G is the star $K_{1,n-1}$.*

Lemma 2.3 ([**6**]). *Let G be a graph with n vertices. Then $q_1(G) \leq d_1(G) + d_2(G)$ with equalsity if and only if G is a regular graph or $G = K_{1,n-1}$.*

For any two graphs G and H, by $\mathcal{N}_G(H)$, we denote the number of subgraphs of G being isomorphic to H. For instance, $\mathcal{N}_G(C_3)$ is the number of triangles of G.

Lemma 2.4 ([**4, 11**]). *Let G be a graph with n vertices, m edges, $\mathcal{N}_G(C_3)$ triangles, and* $\deg(G) = (d_1, \ldots, d_n)$. *Let $T_k = \sum_{i=1}^{n} q_i^k$, $(k = 0, 1, 2, \ldots, n)$ be the k-th spectral moment of the Q-spectrum of G. Then*

$$T_0 = n, \qquad T_1 = \sum_{i=1}^{n} d_i = 2m,$$

$$T_2 = 2m + \sum_{i=1}^{n} d_i^2, \qquad T_3 = 6\mathcal{N}_G(C_3) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

The following lemma is a consequence of Lemma 2.4.

Lemma 2.5 ([**15**]). *Let H be a graph Q-cospectral to G. Then*

(i) *G and H have the same number of vertices.*

(ii) *G and H have the same number of edges.*

(iii) $\sum_{i=1}^{n} d_i^2(G) = \sum_{i=1}^{n} d_i^2(H)$.

(iv) $6\mathcal{N}_G(C_3) + \sum_{i=1}^{n} d_i^3(G) = 6\mathcal{N}_H(C_3) + \sum_{i=1}^{n} d_i^3(H)$.

Since $\mathrm{tr}(A(G)^3) = 6\mathcal{N}_G(C_3)$, from Lemma 2.5, we have the following corollary.

Corollary 2.6 ([**15**]). *If G and H are Q-cospectral and have the same degree of sequences, then $\mathcal{N}_G(C_3) = \mathcal{N}_H(C_3)$.*

Note that

$$q_3'(G) = \mathrm{tr}(A(G)^3) + \sum_{i=1}^{n} (d_i - 2)^3$$

is a graph invariant. The following lemma shows that $q_3'(G)$ is also a Q-cospectral invariant.

Lemma 2.7 ([**15**]). *If G and H are Q-cospectral, then $q_3'(G) = q_3'(H)$.*

Lemma 2.8 ([**4**]). *Let $q_1(G)$ be the spectral radius of the signless Laplacian matrix of G. Then*

(i) $q_1(G) = 0$ *if and only if G has no edges,*
(ii) $0 < q_1(G) < 4$ *if and only if all components of G are paths,*
(iii) *For a connected graph G, we have $q_1(G) = 4$ if and only if G is a cycle C_n or the star graph $K_{1,3}$.*

Recall that for any graph G, $q_n(G) \geq 0$.

Lemma 2.9 ([**4**]). *In any graph G, the multiplicity of the eigenvalue 0 in the Q-spectrum is equal to the number of bipartite components of G.*

Let $S(G)$ be the subdivision graph of G obtained by replacing each edge of G by a path of length two. The lemma below gives the relation between the Q-polynomial of G and the A-polynomial of its subdivision graph $S(G)$.

Lemma 2.10 ([**14**]). *Let G be a graph of order n and size m. Then*

$$\varphi(S(G), \lambda) = \lambda^{m-n}\psi(G, \lambda^2).$$

Lemma 2.11 ([**12**]). *The adjacency eigenvalues of the path P_n are as follows:*

$$\lambda_i = 2\cos\frac{\pi i}{n+1}, \qquad i = 1, 2, \ldots, n.$$

Lemma 2.12 ([**8**]). *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices. Then*

$$\psi(G_1 \bigtriangledown G_2, x) = \frac{\psi(G_1, (x-n_2))\psi(G_2, (x-n_1))}{(x-2r_1-n_2)(x-2r_2-n_1)} f(x),$$

where $f(x) = x^2 - (2(r_1 + r_2) + (n_1 + n_2))x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.

Note that connected graphs with A-index (the largest eigenvalue of the adjacency matrix) less than 2, are proper subgraphs of the Smith graphs (namely, those graphs whose A-index equals 2; see [**5, 10**])

Lemma 2.13 ([**5, 10**]). *Let Π_A^2 denote the set of connected graphs whose A-index is strictly less than 2. Then*

$$\Pi_A^2 = \{P_n \mid n \geq 1\} \cup \{T_{1,1,n-3} \mid n \geq 4\} \cup \{T_{1,2,k} \mid k = 2, 3, 4\},$$

where $T_{a,b,c}$ is a starlike tree depicted in Figure 1.

Lemma 2.14 ([**16**]). *If two graphs G and H are Q-cospectral, then their line graphs are A-cospectral. The converse is true if G and H have the same number of vertices and edges.*

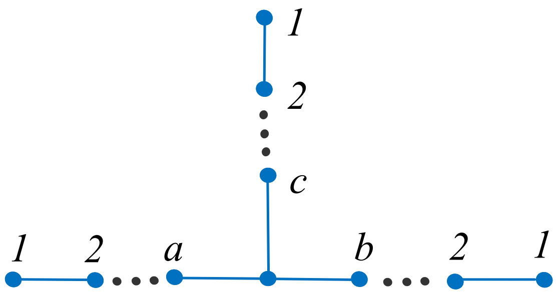

Figure 1. The tree $T_{a,b,c}$ in Lemma 2.13.

3. Main results

Lemma 3.1. *For a path-friendship graph* $\Gamma = G_{s,t_1,\dots,t_k}$*, we have* $q_2(\Gamma) < 4$. *Moreover,*

(i) *If* $t_i \geq 1$ *for each* $1 \leq i \leq k$*, then* $2s + k + 1 \leq q_1(\Gamma) \leq 2s + k + 2$.
(ii) *If* $t_i = 0$ *for each* $1 \leq i \leq k$*, then*

$$q_n(\Gamma) = 1, \qquad q_1(\Gamma) = \frac{2s + 3 + \sqrt{(2s+3)^2 - 16s}}{2}.$$

Proof. Let v be the vertex of Γ with maximum degree. Then

$$\Gamma - v = sK_2 \cup P_{t_1} \cup \dots \cup P_{t_k}.$$

By Lemma 2.11, $\lambda_1(S(\Gamma) - v) < 2$ and by interlacing theorem, $\lambda_2(S(\Gamma)) < 2$. Finally, by Lemma 2.10, we have $q_2(\Gamma) < 4$.

(i) If $t_i \geq 1$ for each $1 \leq i \leq k$, then $d_1(\Gamma) = d(v) = 2s + k$ and $d_2(\Gamma) = 2$, by Lemmas 2.2 and 2.3, we get

$$2s + k + 1 \leq q_1(\Gamma) \leq d_1(\Gamma) + d_2(\Gamma) = 2s + k + 2.$$

(ii) If $t_i = 0$ for each $1 \leq i \leq k$, then Γ is nothing but the friendship graph $F_s = sK_2 \bigtriangledown K_1$. Therefore, by Lemma 2.12, the proof is straightforward. □

Lemma 3.2. *Let* H *be* Q*-cospectral with* $\Gamma = G_{s,t_1,\dots,t_k}$. *Then* H *is a connected graph.*

Proof. Assume that $H = H_1 \cup \dots \cup H_y$, where H_i $(1 \leq i \leq y)$ is a connected component of H. Since Γ is a non-bipartite graph, by Lemma 2.9, any of H_i' is non-bipartite. In view of the fact that $\psi(H) = \prod_{i=1}^{y} \psi(H_i) = \psi(\Gamma)$, by Lemma 2.10, we obtain that

$$\varphi(S(H)) = \prod_{i=1}^{y} \varphi(S(H_i)) = \varphi(S(\Gamma)),$$

implying that there exists some component, say H_1, such that

$$\lambda_1(S(H)) = \lambda_1(S(H_1)) = \lambda_1(S(\Gamma)).$$

On the other hand, by the proof of Lemma 3.1(i), we get

$$\lambda_2(S(H)) = \max\{\lambda_2(S(H_1)),\ \lambda_1(S(H_i)) \mid 2 \le i \le y\} = \lambda_2(S(\Gamma)) < 2.$$

Therefore, by Lemma 2.13 we get $S(H_i) \in \Pi_A^2$. Hence $S(H_i)$ is a tree and H_i is also a tree, for $2 \le i \le y$, a contradiction. □

Let G be a connected graph with n vertices and m edges. If $m = n+k-1$, then G is k-cyclic graph. Clearly, $\Gamma = G_{s,t_1,\ldots,t_k}$ is a s-cyclic graph, where s denotes the number of triangles.

Corollary 3.3. *Let H be Q-cospectral with $\Gamma = G_{s,t_1,\ldots,t_k}$. Then H has at most s triangles.*

Proof. From Lemma 3.2, H is a connected graph. Since H and Γ are Q-cospectral, they have the same number of vertices and the same of number of edges. Therefore, H is a s-cyclic graph. This means that H consists of s cycles. Therefore, H has at most s triangles. □

Lemma 3.4. *Suppose that H and $\Gamma = G_{s,t_1,\ldots,t_k}$ are Q-cospectral. Then the degree sequence of H is determined by the shared Q-spectrum.*

Proof. By Lemma 3.2, H is a connected graph. In addition, it follows from Lemma 2.1 that $d_2(H) \le 4$ since $q_2(H) < 4$. Also, since H and Γ are Q-cospectral, by Lemma 2.5, they have the same order, size and the sum of the squares of degrees of vertices. Assume that H has n_i vertices of degree i for $i = 1, 2, \ldots, d_1(H)$. Therefore,

$$\sum_{i=1}^{d_1(H)} n_i = n(\Gamma), \tag{1}$$

$$\sum_{i=1}^{d_1(H)} i n_i = 2m(\Gamma), \tag{2}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = n'_1 + 4n'_2 + d_1^2(\Gamma), \tag{3}$$

where n'_i $(i = 1, 2)$ is the number of vertices of degree i belonging to Γ. Clearly, $n(\Gamma) = n$, $m(\Gamma) = n + s - 1$, ${n'}_1 = k$, ${n'}_2 = n - (k + 1)$, and $d_1(\Gamma) = 2s + k$. By summing up equations (1), (2), and (3) with coefficients 2, -3, 1, respectively, we get

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2) n_i = 4s^2 + 4sk - 6s + k^2 - 3k + 2. \tag{4}$$

Since $q_1(H) = q_1(\Gamma)$, by Lemma 3.1(i), $2s + k + 1 \le q_1(\Gamma) \le 2s + k + 2$. It follows from Lemma 2.2 that $d_1(H) \le 2s + k + 1$. On the other hand, by Lemma 2.3, we obtain that $2s+k+1 \leq q_1(\Gamma) \leq d_1(H)+d_2(H) \leq d_1(H)+4$, that is, $2s+k-3 \leq d_1(H)$. Consequently, $2s+k-3 \leq d_1(H) \leq 2s+k+1$. It follows from Lemma 2.7 that

$$6\mathcal{N}_H(C_3) + \sum_{i=1}^{n}(d_i(H)-2)^3 = q_3(H) = q_3(\Gamma) = 6s + (2s+k-2)^3 - k \tag{5}$$

or

$$\mathcal{N}_H(C_3) = \frac{1}{6}\Big(6s + (2s+k-2)^3 - k - \sum_{i=1}^{n}(d_i(H)-2)^3\Big). \tag{6}$$

Let us consider the following cases:

Case 1 $d_1(H) = 2s+k-3$. If $n_{2s+k-3} \geq 2$, then

$$(s,k) \in \Big\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3)\Big\}$$

because $2s+k-3 = d_1(H) = d_2(H) \leq 4$. So, let us consider the following subcases:

Case 1-a $(s,k) = (1,1)$. Then $d_1(H) = d_2(H) = 0$, a contradiction, since H is a connected graph.

Case 1-b $(s,k) = (1,2)$. Then $d_1(H) = d_2(H) = 1$. So $H = P_2$ and $\Gamma = P_2$. On the other hand, $d_1(\Gamma) = 4$, an impossibility.

Case 1-c $(s,k) = (1,3)$. Then $d_1(H) = d_2(H) = 2$, and so $n_1 = 0$ and $n_2 = n$. This means that H is a connected 2-regular graph. On the other hand, $\frac{2(n+s-1)}{n} = 2$ or $s = 1$, and so H consists of a triangle, and since H is a regular graph, so $H = C_3 = \Gamma$. On the other hand, if $(s,k) = (1,3)$, then $|V(H)| \geq 5$ (since $d_1(\Gamma) = 5$), a contradiction.

Case 1-d $(s,k) = (1,4)$. Then $d_1(H) = d_2(H) = 3$. So by equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 10, \\ n_2 = n - 20, \\ n_3 = 10. \end{cases}$$

By Lemma 2.14, $\mathcal{L}(\Gamma)$ and $\mathcal{L}(H)$ are A-cospectral, and so the number of triangles of them are the same. Therefore, $20 = \binom{6}{3} = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \mathcal{N}_{\mathcal{L}(H)}(C_3) = 10\binom{3}{3} = 10$, a contradiction.

Case 1-e $(s,k) = (1,5)$. Then $d_1(H) = d_2(H) = 4$. So by equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 15 - n_4, \\ n_2 = n + 3n_4 - 30, \\ n_3 = 15 - 3n_4. \end{cases}$$

It follows from Lemma 2.14 that $\mathcal{L}(\Gamma)$ and $\mathcal{L}(H)$ are A-cospectral and so the number of triangles of them are the same. Therefore,

$$15 + n_4 = (15 - 3n_4)\binom{3}{3} + n_4\binom{4}{3} = \mathcal{N}_{\mathcal{L}(H)}(C_3) = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \binom{7}{3} = 35$$

and as a result $n_4 = 20$, a contradiction, since $n_1 < 0$.

Case 1-f $(s,k) = (2,1)$. Then $d_1(H) = d_2(H) = 2$. So

$$\begin{cases} n_1 = -2, \\ n_2 = n + 2. \end{cases}$$

A contradiction.

Case 1-g $(s,k) = (2,2)$. Then $d_1(H) = d_2(H) = 3$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 8, \\ n_2 = n - 18, \\ n_3 = 10. \end{cases}$$

It follows from equation (6) that $\mathcal{N}_H(C_3) = 12$. It follows from Corollary 3.3 that $12 \leq 2$, a contradiction.

Case 1-h $(s,k) = (2,3)$. Then $d_1(H) = d_2(H) = 4$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 13 - n_4, \\ n_2 = n + 3n_4 - 28, \\ n_3 = 15 - 3n_4. \end{cases}$$

By Lemma 2.14, $\mathcal{L}(\Gamma)$, and $\mathcal{L}(H)$ are A-cospectral and so the number of triangles of them are the same. Therefore,

$$15 + n_4 = (15 - 3n_4)\binom{3}{3} + n_4\binom{4}{3} = \mathcal{N}_{\mathcal{L}(H)}(C_3) = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \binom{7}{3} = 35,$$

and as a result $n_4 = 20$, a contradiction, since $n_1 < 0$. So we can deduce that $n_{2s+k-3} = 1$.

Combining equations (1), (2), and (4), we obtain

$$\begin{cases} n_1 = 6s + 4k - 12 - n_4, \\ n_2 = -12s + 3n_4 + n - 7k + 20, \\ n_3 = 6s - 3n_4 + 3k - 9. \end{cases}$$

By Lemma 2.14, $\mathcal{L}(\Gamma)$ and $\mathcal{L}(H)$ are A-cospectral, and so the number of triangles of them are the same. Therefore,

$$4n_4 + n_3 + \binom{2s+k-3}{3} = \mathcal{N}_{\mathcal{L}(H)}(C_3) = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \binom{2s+k}{3},$$

and so

$$n_4 = 6s^2 + 6ks - 21s + 1.5k^2 - 10.5k + 19.$$

It is easy to see that $n_4 \leq 2s + k - 3$, since $n_3 \geq 0$. Therefore, $f(s,k) = (6s^2 - 23s + 22) + (1.5k^2 - 11.5k + 6ks) \leq 0$. Obviously, for any natural number s, we always have $6s^2 - 23s + 22 \geq 0$. On the other hand, if $s \geq 2$, then $6s^2 - 23s + 22 \geq 0$. By n_{2s+k-3} and $d_1(H) = 2s + k - 3$, we can assume that for $s = 1$, we have $k \geq 2$, and so $1.5k^2 - 11.5k + 6ks > 0$, which implies that $f(s,k) > 0$, a contradiction.

Case 2 $d_1(H) = 2s + k - 2$. If $n_{2s+k-2} \geq 2$, then $(s, k) \in \left\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2)\right\}$ because $2s + k - 2 = d_1(H) = d_2(H) \leq 4$. So, let us consider the following subcases:

Case 2-a $(s, k) = (1, 1)$. Then $d_1(H) = d_2(H) = 1$, and since H is connected, $H = K_2$. On the other hand, if $(s, k) = (1, 1)$, then $|V(H)| \geq 4$, a contradiction.

Case 2-b $(s, k) = (1, 2)$. Then $d_1(H) = d_2(H) = 2$. Now if $d_3(H) = \cdots = d_n(H) = 2$, then H is a connected 2-regular graph. On the other hand, $\frac{2(n+s-1)}{n} = 2$ or $s = 1$. This means that $H = C_3$. On the other hand, if $(s, k) = (1, 2)$, then $|V(H)| \geq 5$, a contradiction. So, the vertex degrees of H are either 2 or 1. In other words, $H = P_n$. By equation (5), $-2 = 12$, a contradiction (Note that any path has exactly two vertices of degree 1).

Case 2-c $(s, k) = (1, 3)$. Then $d_1(H) = d_2(H) = 3$. So by equation (4), $n_3 = 6$, and by equations (1) and (2), $n_2 = n - 12$ and $n_1 = 6$. It follows from equation (6) that $\mathcal{N}_H(C_3) = 5$, a contradiction to Corollary 3.3.

Case 2-d $(s, k) = (1, 4)$. Then $d_1(H) = d_2(H) = 4$. So by equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 10 - n_4, \\ n_2 = n + 3n_4 - 20, \\ n_3 = 10 - 3n_4. \end{cases}$$

By equation (6), we have $\mathcal{N}_H(C_3) = 11 - n_4$. It follows from Corollary 3.3 that $11 - n_4 \leq 1$ or $n_4 \geq 10$. This means that $n_3 < 0$, a contradiction.

Case 2-e $(s, k) = (2, 1)$. Then $d_1(H) = d_2(H) = 3$. So

$$\begin{cases} n_1 = 4, \\ n_2 = n - 10, \\ n_3 = 6. \end{cases}$$

By equation (6), $\mathcal{N}_H(C_3) = 6$. It follows from Corollary 3.3 that $6 \leq 2$, a contradiction.

Case 2-f $(s, k) = (2, 2)$. Then $d_1(H) = d_2(H) = 4$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 8 - n_4, \\ n_2 = n + 3n_4 - 18, \\ n_3 = 10 - 3n_4. \end{cases}$$

It follows from equation (6) that $\mathcal{N}_H(C_3) = 12 - n_4$. It follows from Corollary 3.3 that $n_4 \geq 10$, which means that $n_1 < 0$, a contradiction. Therefore, $n_{2s+k-2} = 1$. Obviously $(s, k) \neq (1, 1)$, otherwise $n_1 = 1$, and so $n_4 = 0$ and as a result, we obtain that $n_1 = 0$, which is impossible. Also, $(s, k) \neq (1, 2)$. Otherwise, $n_2 = 1$, and so $d_1(H) = 2$, which means that $H = P_3$, and so $\Gamma = P_3$. Therefore, $d_1(\Gamma) = 4$, which is impossible. Combining equations (1), (2), and (4), we obtain

$$\begin{cases} n_1 = 4s + 3k - 7 - n_4, \\ n_2 = n - 8s - 5k + 3n_4 + 11, \\ n_3 = 4s + 2k - 3n_4 - 5. \end{cases}$$

Obviously, $\dfrac{8s - n + 5k - 11}{3} \leq n_4 \leq \dfrac{4s - 5 + 2k}{3}$. On the other hand, by equation (6) and Corollary 3.3,

$$n_4 \geq 4s^2 - 12s + 4sk + k^2 - 6k + 9.$$

We claim that

$$4s^2 - 12s + 4sk + k^2 - 6k + 9 > \frac{4s - 5 + 2k}{3}$$

or

$$(12s^2 - 40s + 12sk) + (3k^2 - 20k + 32) > 0.$$

To prove this claim, set $g(s,k) = 12s^2 - 40s + 12sk$ and $f(k) = 3k^2 - 20k + 32$. It is easy to see that $f(3) < 0$, $f(4) = 0$, and $f(k) > 0$ otherwise, and $g(s,k) < 0$ if $(s,k) = (2,1)$ and $g(s,k) > 0$, otherwise. Therefore, for $k \geq 2$, $g(s,k) + f(k) > 0$. Now, we determine the sign of the function $g(s,k) + f(k)$ for $k \in \{1,2,3,4\}$. Obviously, $g(s,3) + f(3) = 12s^2 - 4s - 1 > 0$ and $g(s,4) + f(4) = 12s^2 + 8s > 0$. Clearly $g(2,1) + f(1) = 7$. Therefore, the claim is proved.

By a direct calculation, we obtain that $\mathcal{N}_H(C_3) = 4s^2 - 11s + 4sk + k^2 - 6k + 9 - n_4$. By Corollary 3.3, we obtain that

$$4s^2 - 11s + 4sk + k^2 - 6k + 9 - n_4 \leq s$$

or

$$n_4 \geq 4s^2 - 12s + 4sk + k^2 - 6k + 9 > \frac{4s - 5 + 2k}{3} \geq n_4,$$

which is a contradiction.

Case 3 $d_1(H) = 2s+k-1$. If $n_{2s+k-1} \geq 2$, then $2s+k-1 = d_1(H) = d_2(H) \leq 4$. Therefore,

$$(s,k) \in \Big\{(1,1), (1,2), (1,3), (2,1)\Big\}.$$

Let us consider the following subcases:

Case 3-a $(s,k) = (1,1)$. Then $d_1(H) = d_2(H) = 2$. Now, if $d_3(H) = \cdots = d_n(H) = 2$, then H is a connected 2-regular graph. Moreover, $\frac{2(n+s-1)}{n} = 2$ or $s = 1$. This means that $H = C_3$. On the other hand, if $(s,k) = (1,1)$, then $|V(H)| \geq 4$, a contradiction. So, the vertex degrees of H are either 2 or 1. In other words, $H = P_n$. By equation (6), $-2 = 6$, a contradiction.

Case 3-b $(s,k) = (1,2)$. Then $d_1(H) = d_2(H) = 3$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 3, \\ n_2 = n - 6, \\ n_3 = 3. \end{cases}$$

It follows from equation (6) that $\mathcal{N}_H(C_3) = 2$. It follows from Corollary 3.3 that $2 \leq 1$, a contradiction.

Case 3-c $(s,k) = (1,3)$. Then $d_1(H) = d_2(H) = 4$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 6 - n_4, \\ n_2 = n - 12 + 3n_4, \\ n_3 = 6 - 3n_4. \end{cases}$$

It follows from equation (6) that $\mathcal{N}_H(C_3) = 5 - n_4$. It follows from Corollary 3.3 that $n_4 \geq 4$, a contradiction, since $n_3 < 0$.

Case 3-d $(s,k) = (2,1)$. Then $d_1(H) = d_2(H) = 4$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 4 - n_4, \\ n_2 = n - 10 + 3n_4, \\ n_3 = 6 - 3n_4. \end{cases}$$

It follows from equation (6) that $\mathcal{N}_H(C_3) = 6 - n_4$. It follows from Corollary 3.3 that $n_4 \geq 4$, a contradiction, since $n_3 < 0$. Hence $n_{2s+k-1} = 1$. By equations (1), (2), and (4), we get

$$\begin{cases} n_1 = 2k + 2s - 3 - n_4, \\ n_2 = n - 3k - 4s + 4 + 3n_4, \\ n_3 = k + 2s - 2 - 3n_4. \end{cases}$$

It follows from equation (6) that

$$\mathcal{N}_H(C_3) = \frac{4s^2 - 8s + 4sk + k^2 - 5k + 6 - 2n_4}{2}.$$

It follows from Corollary 3.3 that

$$\frac{4s^2 - 8s + 4sk + k^2 - 5k + 6 - 2n_4}{2} \leq s.$$

We conclude that $4s^2 - 10s + 4sk + k^2 - 5k + 6 - 2n_4 \leq 0$, and so

$$2k + 2s - 3 \geq n_4 \geq 2s^2 - 5s + 2sk + 3 + \frac{k^2 - 5k}{2}$$

or

$$2s^2 - 7s + 2sk + 6 + \frac{k^2 - 9k}{2} \leq 0.$$

Thus

$$f(s,k) = 4s^2 - 14s + 4sk + 12 + k^2 - 9k \leq 0.$$

For $s \geq 3$, $f(s,k) > 0$, a contradiction. Consider the following subcases:

1. $s = 1$.
   - (a) $k = 1$. Then $n_2 = 1$ $(d_1(H) = 2)$, which implies that $n_3 = n_4 = 0$. On the other hand, $n_3 = 1$, which is impossible.
   - (b) $k = 2$. Then $n_3 = 1$. As a result $n_4 = 0$. On the other hand, for $(s,k) = (1,2)$, $n_3 = 2$, a contradiction.

(c) $k = 3$. Then $n_4 = 1$. So, $n_3 = 0$. By Lemma 2.14, $\mathcal{L}(\Gamma)$, and $\mathcal{L}(H)$ are A-cospectral and so the number of triangles of them are the same. Therefore, $10 = \binom{5}{3} = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \mathcal{N}_{\mathcal{L}(H)}(C_3) = \binom{4}{3} = 4$, a contradiction.

(d) $k = 4$. Then $n_5 = 1$. As a result $n_3 = 4 - 3n_4$. If $n_4 = 0$, then $n_3 = 4$. Lemma 2.14 implies that $20 = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \mathcal{N}_H(\Gamma) = 14$, a contradiction. If $n_4 = 1$, then $n_3 = 1$. It follows from Lemma 2.14 that $20 = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \mathcal{N}_{\mathcal{L}(H)}(C_3) = 17$, which is impossible.

2. $s = 2$.

(a) For $k \geq 5$, $f(s, k) > 0$, a contradiction.

(b) $k = 1$. Then $n_4 = 1$, and so $n_3 = 0$. It follows from Lemma 2.14 that $10 = \mathcal{N}_{\mathcal{L}(\Gamma)}(C_3) = \mathcal{N}_{\mathcal{L}(H)}(C_3) = 4$, a contradiction.

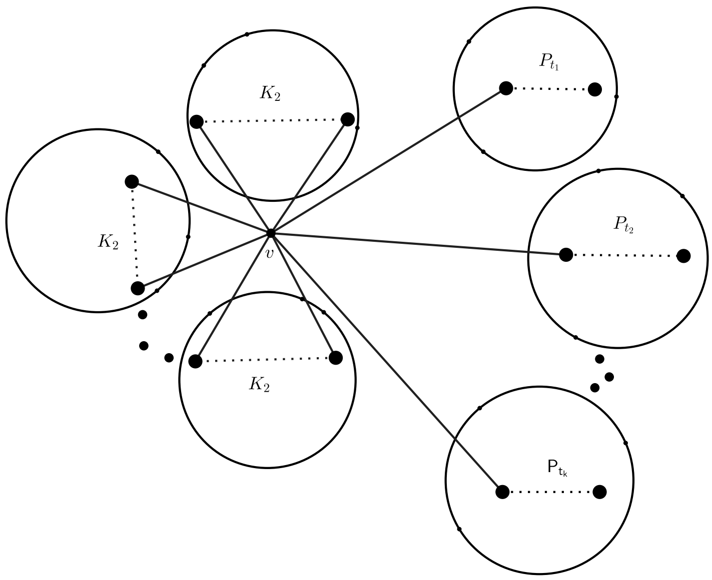

Figure 2. The path-friendship graph $G_{s,t_1,\ldots,t_k}$ and its connected components after removing the vertex v.

<u>Case 4</u> Suppose that $d_1(H) = 2s + k + 1$. If $n_{2s+k+1} \geq 2$, then $2s + k + 1 = d_1(H) = d_2(H) \leq 4$. Therefore, $(s, k) = (1, 1)$. By equation (4), we get $n_3 + 3n_4 = 1$. On the other hand, since $n_4 \geq 2$, so $1 = n_3 + 3n_4 \geq n_3 + 6$ or $n_3 \leq -5$, a contradiction. Therefore, $n_{2s+k+1} = 1$, then by equation (4), $0 \leq n_3 + 3n_4 = -2s - k + 2 < 0$, a contradiction. So, this case cannot happen.

<u>Case 5</u> Suppose that $d_1(H) = 2s + k$. If $n_{2s+k} \geq 2$, then $2s + k = d_1(H) = d_2(H) \leq 4$. Therefore, $(s, k) \in \{(1, 1), (1, 2)\}$. If $(s, k) = (1, 1)$, then $d_1(H) =$

$d_2(H) = 3$, and so $n_3 \geq 2$. By equation (4), we get $n_3 = 1$, a contradiction. If $(s, k) = (1, 2)$, then $d_1(H) = d_2(H) = 4$. By equation (4), we get $n_3 + 3n_4 = 3$. On the other hand, since $n_4 \geq 2$, so $3 = n_3 + 3n_4 \geq n_3 + 6$ or $n_3 \leq -3$, a contradiction. Therefore, $n_{2s+k} = 1$. By equation (4), $n_3 = n_4 = 0$ and by equations (1) and (2), $n_1 = k$ and $n_2 = n - (k + 1)$. Also, it is clear that in this case, $\mathcal{N}_H(C_3) = s$.

Hence, it is proved that $\deg(H) = \deg(\Gamma)$. □

Lemma 3.5. *Any graph H Q-cospectral with a path-friendship graph $\Gamma = G_{s,t_1,\dots,t_k}$ is a path-friendship graph.*

Proof. By Lemma 3.4, H has a unique vertex v that $d_H(v) = 2s + k \geq 4$, and also H has exactly s triangles. Note that these s triangles are common in v (v is the vertex with maximum degree belonging to H). Because $d_H(v) = 2s + k$, $n_{2s+k} = 1$, and the vertex degrees of H belong to $\{1, 2, 2s + k\}$. By removing the vertex v, $H - v = sK_2 \cup G_1 \cup \cdots \cup G_h$, where h is a natural number. In addition, the vertex degrees of G_i belong to $\{0, 1, 2\}$, and so every G_i is either a path or a cycle. By contradiction, suppose that $G_1 = C_k$, where $k \geq 3$ is a natural number. Therefore, H must have a vertex with the degree greater than two, a contradiction, since the vertex degrees of H belong to $\{1, 2, 2s + k\}$ (Note that $d_H(v) = 2s + k$ and $n_{2s+k} = 1$). So, none of G_i's ($1 \leq i \leq k$) are cycles. Since H has s triangles and the vertex degrees of H belong to $\{1, 2, 2s + k\}$. Hence, it is proved that H is also a path-friendship graph (see Figure 2). □

Before proving the main result, we state an essential lemma.

Lemma 3.6 ([7])**.** *Suppose G is a nontrivial simple connected graph. Let u be a vertex of G. For nonnegative integers k and l, let $G(k, l)$ denote the graph obtained from G by adding pendant paths of length k and l at u. If $k \geq l \geq 1$, then*

$$q_1(G(k, l)) > q_1(G(k + 1, l - 1)).$$

Theorem 3.7. *All path-friendship graphs are* DQS.

Proof. Let H be any graph Q-cospectral with $G_{s,t_1,\dots,t_k} = \Gamma$. By Lemma 3.5, H is a path-friendship graph. Therefore, $H = G_{s,l_1,\dots,l_k}$. If H and Γ are non isomorphic, then there exists some $1 \leq i \leq k$, $t_i \neq l_i$. It follows from Lemma 3.6 that $q_1(H) \neq q_1(\Gamma)$, which is impossible, since H and Γ are Q-cospectral. □

Acknowledgment. The authors would like to thank anonymous referees whose valuable comments and suggestions improved the final version of the paper.

References

1. Abdian A. Z., Ashrafi A. R. and Brunetti M., *Signless Laplacian spectral characterization of roses*, Kuwait J. Sciences **47** (2020), 12–18.

2. Bu C. and Zhou J., *Starlike trees whose maximum degree exceed 4 are determined by their Q-spectra*, Linear Algebra Appl. **436** (2012), 143–151.

3. Cvetković D., Rowlinson P. and Simić S., *Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math. (Beograd) **81** (2007), 11–27.

4. Cvetković D., Rowlinson P. and Simić S., *Signless Laplacians of finite graphs*, Linear Algebra Appl. **423** (2007), 155–171.
5. Cvetković D., Doob M. and Sachs H., *Spectra of Graphs-Theory and Applications*, III revised and enlarged ed., Johan Ambrosius Bart Verlag, Heidelberg, Leipzig, 1995, (2013), 19–25.
6. Das K. Ch., *On conjectures involving second largest signless Laplacian eigenvalue of graphs*, Linear Algebra Appl. **432** (2010), 3018–3029.
7. Feng L., *The signless Laplacian spectral radius for bicyclic graphs with k pendant vertices*, Kyungpook Math. J. **50** (2010), 109–116.
8. Liu X. and Lu P., *Signless Laplacian spectral characterization of some joins*, Electron. J. Linear Algebra. **30** (2015), #30.
9. Omidi G .R. and Vatandoost E., *Starlike trees with maximum degree 4 are determined by their signless Laplacian spectra*, Electron. J. Linear Algebra **20** (2010), 274–290.
10. Smith J. H., *Some properties of the spectrum of a graph*, in: Combinatorial Structures and their Applications, Proc. Conf. Calgary, 1969 (R. Guy, et al. eds.), Gordon and Breach, New York, 1970, 403–406.
11. Simić, S. K. and Stanić Z., *Q-integral graphs with edge-degrees at most five*, Discrete Math. **308** (2008), 4625–4634.
12. van Dam E. R. and Haemers W. H., *Which graphs are determined by their spectrum?*, Linear Algebra Appl. **373** (2003), 241–272.
13. Wang J. F., Belardo F., Huang Q. X. and Boroviánin B., *On the two largest Q-eigenvalues of graphs*, Discrete Math. **310** (2010), 2858–2866
14. Wang J. F., Huang Q. X., Belardo F. and Li Marzi E. M., *On graphs whose signless Laplacian index does not exceed 4.5*, Linear Algebra Appl. **431** (2009), 162–178.
15. Wen F., Huang Q., Huang X. and Liu F., *The spectral characterization of wind-wheel graphs*, Indian J. Pure Appl. Math. **46** (2015), 613–631.
16. Zhou J. and Bu C., *Spectral characterization of line graphs of starlike trees*, Linear and Multilinear Algebra **61** (2013), 1041–1050.

R. Sharafdini, Department of Mathematics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr 75169, Iran,
e-mail: `sharafdini@pgu.ac.ir`

A. Z. Abdian, Department of Mathematical Sciences, Lorestan University, College of Science, Lorestan, Khoramabad, Iran,
e-mail: `abdian.al@fs.lu.ac.ir`