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PELL AND PELL-LUCAS NUMBERS AS SUMS OF THREE REPDIGITS

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ABSTRACT. In this paper, we find all Pell and Pell-Lucas numbers expressible as sums of three base 10 repdigits.

1. INTRODUCTION

Let $\{P_m\}_{m\geq 0}$ be the Pell sequence given by

$$P_{m+2} = 2P_{m+1} + P_m, \tag{1}$$

for $m \geq 0$, where $P_0 = 0$ and $P_1 = 1$. Its first few terms are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, \ldots$$

The Binet formula for its general term is

$$P_m = \frac{\alpha^m - \beta^m}{2\sqrt{2}} \qquad \text{for all } m \geq 0, \tag{2}$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are the two roots of the characteristic equation $x^2 - 2x - 1 = 0$.

Let $\{Q_m\}_{m\geq 0}$ be the companion Lucas sequence of the Pell sequence also called the sequence of Pell–Lucas numbers. It starts with $Q_0 = 2$, $Q_1 = 2$, and obeys the same recurrence relation

$$Q_{m+2} = 2Q_{m+1} + Q_m \qquad \text{for all } m \geq 0 \tag{3}$$

as the Pell sequence. Its first few terms are

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, 228486, 551614, \ldots$$

Its Binet formula is

$$Q_m = \alpha^m + \beta^m \qquad \text{for all } \; m \geq 0. \tag{4}$$

In this paper, we study the Diophantine equations

$$E_n = d_1\Big(\frac{10^{m_1} - 1}{9}\Big) + d_2\Big(\frac{10^{m_2} - 1}{9}\Big) + d_3\Big(\frac{10^{m_3} - 1}{9}\Big), \qquad E \in \{P, Q\}, \tag{5}$$

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in integers $n \geq 0$, $1 \leq m_1 \leq m_2 \leq m_3$, and $d_1, d_2, d_3 \in \{1, 2, \ldots, 9\}$.

Here, we prove the following results.

Theorem 1.1. *The largest Pell number which is a sum of three repdigits is*

$$P_9 = 985 = 888 + 88 + 9. \tag{6}$$

Theorem 1.2. *The largest Pell-Lucas number which is a sum of three repdigits is*

$$Q_{10} = 6726 = 6666 + 55 + 5. \tag{7}$$

We organize this paper as follows. In Section 2, we recall a result due to Matveev concerning a lower bound of a linear forms of logarithms of algebraic numbers, and describe a reduction method due to de Weger. The proofs of Theorem 1.1 and Theorem 1.2 are achieved in Sections 3 and 4, respectively. We start with some elementary considerations.

2. Preliminaries

2.1. Linear forms in logarithms

We need some results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. We start by recalling [**5**, Theorem 9.4], which is a modified version of a result of Matveev [**15**]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1, and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|\}$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is in [**5**, Theorem 9.4].

Theorem 2.1. *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$, *then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \ldots A_l.$$

2.2. The de Weger reduction

Here, we present a variant of the reduction method due to de Weger [**16**].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{8}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{9}$$

$$Y \le X \le X_0. \tag{10}$$

When $\beta = 0$ in (8), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta := -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime, and x_1 is positive. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and $x_1 > 0$. We have the following results.

Lemma 2.1 ([**16**, Lemma 3.1]). *If* (9) *and* (10) *hold for* x_1, x_2 *with* $X \ge 1$ *and* $\beta = 0$, *then* $(-x_2, x_1) = (p_k, q_k)$ *for an index* k *that satisfies*

$$k \le -1 + \frac{\log(1 + X_0\sqrt{5})}{\log\left(\frac{1+\sqrt{5}}{2}\right)} := Y_0.$$

Lemma 2.2 ([**16**, Lemma 3.2]). *Let*

$$A = \max_{0 \le k \le Y_0} a_{k+1}.$$

If (9) *and* (10) *hold for* x_1, x_2 *with* $X \ge 1$ *and* $\beta = 0$, *then*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)}{|\vartheta_2|}\right) + \frac{1}{\delta}\log X < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{11}$$

When $\beta \neq 0$ in (8), we put $\psi := \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we use the notation $\|x\| := \min\{|x - n|, n \in \mathbb{Z}\}$ for the distance from x to the nearest integer. We have the following result.

Lemma 2.3 ([**16**, Lemma 3.3]). *If* (9) *and* (10) *hold for* x_1, x_2 *with* $X \ge 1$ *and* $\beta \neq 0$, *and suppose additionally that*

$$\|q\| > \frac{2X_0}{q},$$

then, the solutions of (9) *and* (10) *satisfy*

$$Y < \frac{1}{\delta} \log \Big(\frac{q^2 c}{|\vartheta_2| X_0} \Big).$$

3. The proof of Theorem 1.1

3.1. An elementary estimate

We assume that

$$P_n = d_1 \Big(\frac{10^{m_1} - 1}{9} \Big) + d_2 \Big(\frac{10^{m_2} - 1}{9} \Big) + d_3 \Big(\frac{10^{m_3} - 1}{9} \Big) \tag{12}$$

for some integers $m_1 \le m_2 \le m_3$ and $d_1, d_2, d_3 \in \{1, 2, \ldots, 9\}$. A quick computation with Maple reveals no solutions in the interval $n \in [10, 500]$. For this computation, we first note that P_{500} has 191 digits. Thus, we generate the list of all numbers which are sums of at most 2 repdigits with at most 191 digits each. Let us call the list $\mathcal{A}$. Then, for every $n \in [10, 500]$, we compute $M := \lfloor \log P_n / \log 10 \rfloor + 1$ (the number of digits of P_n), and then check whether $P_n - d \left(\frac{10^m - 1}{9} \right)$ is a member of $\mathcal{A}$ for some digit $d \in \{1, \ldots, 9\}$ and some $m \in \{M - 1, M\}$. This computation takes a few seconds. So, from now on, we assume that $n \ge 500$. We next comment the size of m_1, m_2, m_3 versus n.

Lemma 3.1. *All solutions of equation* (12) *satisfy*

$$m_3 \log 10 - 3 < n \log \alpha < m_3 \log 10 + 3.$$

Proof. The proof follows easily from the fact that $\alpha^{n-2} < P_n < \alpha^{n-1}$. One can see that

$$\alpha^{n-2} < P_n < 3 \cdot 10^{m_3}.$$

Taking the logarithm on both sides, we get $(n-2) \log \alpha < \log 3 + m_3 \log 10$, which leads to

$$n \log \alpha < 2 \log \alpha + \log 3 + m_3 \log 10 < m_3 \log 10 + 3.$$

Similarly, the lower bound follows. □

3.2. Bounds of n, m_1, m_2, m_3

We next return to equation (12) and use the Binet formula (2) to get

$$\frac{\alpha^n - \beta^n}{2\sqrt{2}} = d_1 \Big(\frac{10^{m_1} - 1}{9} \Big) + d_2 \Big(\frac{10^{m_2} - 1}{9} \Big) + d_3 \Big(\frac{10^{m_3} - 1}{9} \Big).$$

Equation (12) can be expressed as

$$\frac{9}{2\sqrt{2}} (\alpha^n - \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} - d_3 10^{m_3} = -(d_1 + d_2 + d_3). \tag{13}$$

We examine (13) in three different steps as follows.

Step 1. Equation (13) gives

$$\frac{9}{2\sqrt{2}}\alpha^n - d_3 10^{m_3} = d_1 10^{m_1} + d_2 10^{m_2} + \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2 + d_3), \tag{14}$$

which we rewrite as

$$\left|\frac{9}{2\sqrt{2}}\alpha^n - d_3 10^{m_3}\right| = \left|d_1 10^{m_1} + d_2 10^{m_2} + \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2 + d_3)\right| < 48 \cdot 10^{m_2}.$$

Thus, dividing both sides by $d_3 10^{m_3}$, we get

$$\left|\left(\frac{9}{2\sqrt{2}d_3}\right)\alpha^n 10^{-m_3} - 1\right| < \frac{48}{10^{m_3-m_2}}. \tag{15}$$

Let

$$\Gamma_1 := \left(\frac{9}{2\sqrt{2}d_3}\right)\alpha^n 10^{-m_3} - 1. \tag{16}$$

Suppose that $\Gamma_1 = 0$. Then, we have

$$\alpha^n = \frac{2\sqrt{2}d_3 10^{m_3}}{9},$$

so $\alpha^{2n} \in \mathbb{Q}$, a contradiction. Thus, $\Gamma_1 \neq 0$. With the notations of Theorem 2.1, we take

$$\eta_1 = \frac{9}{2\sqrt{2}d_3}, \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad b_1 = 1, \quad b_2 = n, \quad b_3 = -m_3.$$

Since $10^{m_3-1} < P_n < \alpha^{n-1}$, we have that $m_3 < n$. Therefore, we can take $D = n$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\sqrt{2})$, so $d_{\mathbb{L}} = 2$. We now need to take A_j for $j = 1, 2, 3$, such that

$$A_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}.$$

Note that

$$h(\eta_1) \leq h(9) + h(2d_3\sqrt{2}) \leq h(9) + h(18) + h(\sqrt{2}).$$

This implies that

$$2h(\eta_1) < 10.9.$$

Thus, we can take

$$A_1 := 10.9.$$

Clearly,

$$h(\eta_2) = \frac{1}{2}\log\alpha, \qquad h(\eta_3) = \log 10.$$

We have

$$\max\{2h(\eta_2), |\log \eta_2|, 0.16\} = \log\alpha < 0.9 := A_2, \tag{17}$$

$$\max\{2h(\eta_3), |\log \eta_3|, 0.16\} = 2\log 10 < 4.7 := A_3. \tag{18}$$

Theorem 2.1 tells us that

$$\log|\Gamma_1| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (15) leads to

$$(m_3 - m_2)\log 10 < \log(48) + 4.48 \cdot 10^{13}(1 + \log n),$$

giving

$$m_3 - m_2 < 2.09 \cdot 10^{13}(1 + \log n). \tag{19}$$

Step 2. Equation (13) becomes

$$\frac{9}{2\sqrt{2}}\alpha^n - d_3 10^{m_3} - d_2 10^{m_2} = d_1 10^{m_1} + \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2 + d_3), \tag{20}$$

which we rewrite as

$$\begin{aligned}\Big|\frac{9}{2\sqrt{2}}\alpha^n - 10^{m_2}(d_3 10^{m_3-m_2} + d_2)\Big| &= \Big|d_1 10^{m_1} + \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2 + d_3)\Big| \\ &< 39 \cdot 10^{m_1}.\end{aligned}$$

Thus, dividing both sides by $10^{m_2}(d_3 10^{m_3-m_2} + d_2)$, we get

$$\Big|\Big(\frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2} + d_2)}\Big)\alpha^n 10^{-m_2} - 1\Big| < \frac{39}{10^{m_2-m_1}}. \tag{21}$$

Let

$$\Gamma_2 := \Big(\frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2} + d_2)}\Big)\alpha^n 10^{-m_2} - 1. \tag{22}$$

Suppose that $\Gamma_2 = 0$. Then, we have

$$\alpha^n = 2\sqrt{2}\Big(\frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}\Big).$$

Conjugating in $\mathbb{Q}(\sqrt{2})$, we get

$$\beta^n = -2\sqrt{2}\Big(\frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}\Big).$$

Consequently, we obtain

$$\frac{10^{m_3}}{9} \le 2\sqrt{2}\Big(\frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}\Big) = |\beta|^n < 1,$$

which is a contradiction. Thus, $\Gamma_2 \neq 0$. To apply Theorem 2.1, we take

$$\eta_1 = \frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2} + d_2)}, \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad b_1 = 1, \quad b_2 = n, \quad b_3 = -m_2.$$

Again we take $D = n$. Furthermore, we have

$$\begin{aligned}h(\eta_1) &= h\Big(\frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2} + d_2)}\Big) \\ &\le h(9) + h(2\sqrt{2}(d_3 10^{m_3-m_2} + d_2)) \\ &\le h(9) + h(\sqrt{2}) + h(2d_3) + h(2d_2) + (m_3 - m_2)h(10) + \log 2 \\ &\le 9 \cdot 4 + 2 \cdot 4(m_3 - m_2).\end{aligned}$$

That is,

$$2h(\eta_1) < 18.8 + 4.8(m_3 - m_2).$$

Thus, we take

$$A_1 = 18.8 + 4.8(m_3 - m_2).$$

Since $\eta_2,\ \eta_3$ are the same as in Γ_1, we use the same values for $A_2,\ A_3$. From Theorem 2.1, we obtain

$$\log|\Gamma_2| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (21) leads to

$$(m_2 - m_1)\log 10 < \log 39 + 4.11 \cdot 10^{12}(18.8 + 4.8(m_3 - m_2))(1 + \log n).$$

Hence, using inequality (19), we obtain

$$(m_2 - m_1)\log 10 - \log 39 < 4.11 \cdot 10^{12}(18.8 + 4.8(2.09 \cdot 10^{13}(1 + \log n)))(1 + \log n).$$

The above inequality gives us

$$m_2 - m_1 < 1.8 \cdot 10^{26}(1 + \log n)^2. \tag{23}$$

Step 3. Equation (13) becomes

$$\frac{9}{2\sqrt{2}}\alpha^n - d_3 10^{m_3} - d_2 10^{m_2} - d_1 10^{m_1} = \frac{9}{2\sqrt{2}}\beta^n - (d_1 + d_2 + d_3), \tag{24}$$

which we rewrite as

$$\begin{aligned}\Big|&\frac{\alpha^n}{2\sqrt{2}} - \frac{10^{m_3}(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3)}{9}\Big| \\ &= \Big|\frac{\beta^n}{2\sqrt{2}} - \frac{(d_1 + d_2 + d_3)}{9}\Big| < 4.\end{aligned}$$

Thus, dividing both sides by $\frac{\alpha^n}{2\sqrt{2}}$, we get

$$\Big|1 - \alpha^{-n} 10^{m_3}(2\sqrt{2}(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3))/9\Big| < \frac{4\alpha^2}{\alpha^n} < \frac{1}{\alpha^{n-3.58}}. \tag{25}$$

Put

$$\Gamma_3 := 1 - \Big(\frac{2\sqrt{2}(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3)}{9}\Big)\alpha^{-n} 10^{m_3}. \tag{26}$$

The fact that $\Gamma_3 \neq 0$ can be justified by a similar argument as the fact that $\Gamma_2 \neq 0$. In order to apply Theorem 2.1, we take

$$\eta_1 = \frac{2\sqrt{2}(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3)}{9}, \quad \eta_2 = \alpha, \quad \eta_3 = 10,$$

$$b_1 = 1, \quad b_2 = -n, \quad b_3 = m_3.$$

We have $D = n$, and A_2 and A_3 are as in (17) and (18). As for A_1, we have

$$\begin{aligned} h(\eta_1) &= h\Big(\frac{2\sqrt{2}(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3)}{9}\Big) \\ &\le h\Big(\frac{2\sqrt{2}(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_2} + d_3)}{9}\Big) \\ &\le h(9) + h(\sqrt{2}) + h(2(d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_2} + d_3)) \end{aligned}$$

$$\begin{aligned}&\leq h(9)+h(\sqrt{2})+h(2d_1)+h(2d_2)+h(2d_3)+(m_3-m_2)h(10)\\&\quad+(m_2-m_1)h(10)+2\log 2\\&\leq 12.95+2.4(m_3-m_2)+2.4(m_2-m_1).\end{aligned}$$

That is,

$$2h(\eta_1)<25.9+4.8(m_3-m_2)+4.8(m_2-m_1).$$

Thus, we can take

$$A_1=25.9+4.8(m_3-m_2)+4.8(m_2-m_1).$$

Theorem 2.1 tells us that

$$\log|\Gamma_4|>-1.4\cdot 30^{l+3}l^{4.5}d_{\mathbb{L}}^2(1+\log d_{\mathbb{L}})(1+\log D)A_1A_2A_3.$$

Comparing this last inequality with (25) leads to

$$n\log\alpha-\log(\alpha^{3.58})<4.11\cdot 10^{12}(25.9+4.8(m_3-m_2)+4.8(m_2-m_1))(1+\log n).$$

Hence, using inequalities (19) and (23), we obtain

$$\begin{aligned}n\log\alpha-\log(\alpha^{3.58})<4.11\cdot 10^{12}(25.9&+4.8(2.09\cdot 10^{13}(1+\log n))\\&+4.8(1.8\cdot 10^{26}(1+\log n)^2))(1+\log n).\end{aligned}$$

The above inequality gives us

$$n<4.83\cdot 10^{45}.$$

Lemma 3.1 implies

$$m_1\leq m_2\leq m_3<2.1\cdot 10^{45}.$$

We summarize what we have proved so far in the following lemma.

Lemma 3.2. *All solutions of equation (12) satisfy*

$$m_1\leq m_2\leq m_3<2.1\cdot 10^{45},\qquad n<4.83\cdot 10^{45}.$$

3.3. Reducing the bound

To lower the above bounds, we return to equation (12). We rewrite it into the form

$$P_n=\frac{d_3 10^{m_3}}{9}+\Big(d_1\frac{10^{m_1}-1}{9}+d_2\frac{10^{m_2}-1}{9}-\frac{d_3}{9}\Big).$$

Observe that the term in parentheses is always positive since

$$\Big(d_1\frac{10^{m_1}-1}{9}+d_2\frac{10^{m_2}-1}{9}-\frac{d_3}{9}\Big)\geq 2\frac{10^{m_1}-1}{9}-\frac{1}{9}\geq 2-\frac{1}{9}\geq\frac{7}{4}>0.$$

Hence, we have

$$\begin{aligned}\frac{\alpha^n}{2\sqrt{2}}-\frac{d_3 10^{m_3}}{9}&=\Big(d_1\frac{10^{m_1}-1}{9}+d_2\frac{10^{m_2}-1}{9}-\frac{d_3}{9}\Big)\\&\quad+\frac{\beta^n}{2\sqrt{2}}\geq\frac{7}{4}-\frac{1}{2\sqrt{2}\alpha^{500}}>0.\end{aligned}$$

Thus, the number Γ_1 from (16) appearing inside the absolute value in inequality (15) is positive. Hence, with the above notations, we have

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{d_3 10^{m_3}}{9} = \frac{d_3 10^{m_3}}{9}\left(\mathrm{e}^{\Lambda_1} - 1\right) > 0.$$

Let

$$\Lambda_1 = n\log\eta_2 - m_3\log\eta_3 + \log\eta_1.$$

Therefore, we obtain

$$0 < \Lambda_1 < \exp(\Lambda_1) - 1 = \Gamma_1 < \frac{48}{10^{m_3-m_2}},$$

which implies that

$$\begin{aligned}0 < \log\left(\frac{9}{2d_3\sqrt{2}}\right) + m_3(-\log 10) + n\log\alpha &< \frac{48}{10^{m_3-m_2}}\\ &< 10^{1.69}\exp(-2.30\cdot(m_3-m_2)).\end{aligned}$$

Thus,

$$\Lambda_1 < 10^{1.69}\exp(-2.30\cdot(m_3-m_2)),$$

with $Y := m_3 - m_2 < 2.1\cdot 10^{45}$.

Therefore, to apply Lemma 2.3, we take

$$c = 10^{1.69}, \qquad \delta = 2.3, \qquad X_0 = 2.1\cdot 10^{45}, \quad \psi = \frac{\log(9/2d_3\sqrt{2})}{\log 10},$$

$$\vartheta = -\frac{\log\alpha}{\log 10}, \quad \vartheta_1 = \log\alpha, \quad \vartheta_2 = \log 10, \qquad \beta = \log(9/2d_3\sqrt{2}).$$

The smallest value of $q > X_0$ is $q = q_{102}$. We find that q_{102} satisfies the hypothesis of Lemma 2.3 for all $d_3 = 1, \ldots, 9$. Applying Lemma 2.3, we get $m_3 - m_2 \le 53$.

We now take $0 \le m_3 - m_2 \le 53$. Let

$$\Lambda_2 = n\log\eta_2 - m_2\log\eta_3 + \log\eta_1.$$

From equation (13), we have

$$\begin{aligned}\frac{d_3 10^{m_3} + d_2 10^{m_2}}{9}(\mathrm{e}^{\Lambda_2} - 1) &= \frac{\beta^n}{2\sqrt{2}} + d_1\frac{10^{m_1}-1}{9} - \left(\frac{d_3+d_2}{9}\right)\\ &> \frac{(-1)^n}{2\sqrt{2}\alpha^n} + \frac{10^{m_1}}{9} - \frac{1}{3}.\end{aligned}$$

Furthermore,

$$\frac{(-1)^n}{2\sqrt{2}\alpha^n} + \frac{10^{m_1}}{9} - \frac{1}{3} > -\frac{1}{2\sqrt{2}\alpha^n} + \frac{7}{9} > -\frac{1}{2\sqrt{2}\alpha^{500}} + \frac{7}{9} > 0.$$

Thus,

$$\mathrm{e}^{\Lambda_2} - 1 > 0.$$

So, from (20), we see that

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{d_3 10^{m_3}}{9} - \frac{d_2 10^{m_2}}{9} = \left(\frac{d_3 10^{m_3}}{9} + \frac{d_2 10^{m_2}}{9}\right)\left(\mathrm{e}^{\Lambda_2} - 1\right) > 0,$$

then by (21),

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = \Gamma_2 < \frac{39}{10^{m_2-m_1}},$$

which implies that

$$\begin{aligned} 0 &< \log\Big(\frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2}+d_2)}\Big) + m_2(-\log 10) + n\log\alpha \\ &< \frac{39}{10^{m_2-m_1}} < 10^{1.6}\exp(-2.30\cdot(m_2-m_1)). \end{aligned}$$

Thus, we get

$$\Lambda_2 < 10^{1.6}\exp(-2.30\cdot(m_2-m_1)),$$

with $Y := m_2 - m_1 < 2.1\cdot 10^{45}$.

Therefore, in order to apply Lemma 2.3, we take

$$c = 10^{1.6}, \qquad \delta = 2.3, \qquad X_0 = 2.1\cdot 10^{45}, \quad \psi = \frac{\log\Big(\frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2}+d_2)}\Big)}{\log 10},$$

$$\vartheta = -\frac{\log\alpha}{\log 10}, \quad \vartheta_1 = \log\alpha, \quad \vartheta_2 = \log 10, \qquad \beta = \log\Big(\frac{9}{2\sqrt{2}(d_3 10^{m_3-m_2}+d_2)}\Big).$$

We get $q = q_{104} > X_0$. By Lemma 2.3 for $d_2 = 1,\ldots,9$, $d_3 = 1,\ldots,9$, and $m_3 - m_2 \le 53$, we get also $m_2 - m_1 \le 55$.

We now take $0 \le m_3 - m_1 \le 108$ and $0 \le m_3 - m_2 \le 53$. Let

$$\Lambda_3 = m_3\log\eta_3 - n\log\eta_2 + \log\eta_1.$$

From equation (13), we have

$$\frac{\alpha^n}{2\sqrt{2}}(1 - e^{\Lambda_3}) = \frac{\beta^n}{2\sqrt{2}} - \frac{d_1+d_2+d_3}{9}.$$

Furthermore,

$$-\frac{\beta^n}{2\sqrt{2}} + \frac{d_1+d_2+d_3}{9} > -\frac{1}{2\sqrt{2}\alpha^n} + \frac{1}{3} > -\frac{1}{2\sqrt{2}\alpha^{500}} + \frac{1}{3} > 0.$$

Thus,

$$e^{\Lambda_3} - 1 > 0.$$

So

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = |\Gamma_3| < \frac{1}{\alpha^{n-3.58}},$$

by (25), which implies that

$$\begin{aligned} 0 &< \log\Big(\frac{2\sqrt{2}(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)}{9}\Big) + m_3\log 10 + n(-\log\alpha) \\ &< \frac{1}{\alpha^{n-3.58}} < \alpha^{3.58}\exp(-0.88\cdot n). \end{aligned}$$

We keep the value for $X_0 = 2.1\cdot 10^{45}$, and only change ψ to

$$\psi = \log\Big(\frac{2\sqrt{2}(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)}{9}\Big)/\log 10,$$

$$c = \alpha^{3.58}, \qquad \delta = 0.88, \qquad v = \frac{\log\alpha}{\log 10},$$
$$v_1 = -\log\alpha, \quad v_2 = \log 10, \quad \beta = \log\Big(\frac{2\sqrt{2}(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)}{9}\Big).$$

We take $q = q_{125} > X_0$ and by Lemma 2.3, we get $n \le 209$. But this contradicts the assumption that $n \ge 500$. Hence, the theorem is proved.

4. The proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We may sometimes omit some details.

4.1. An elementary estimate

We assume that

$$Q_n = d_1\Big(\frac{10^{m_1}-1}{9}\Big) + d_2\Big(\frac{10^{m_2}-1}{9}\Big) + d_3\Big(\frac{10^{m_3}-1}{9}\Big) \tag{27}$$

for some integers $m_1 \le m_2 \le m_3$ and $d_1, d_2, d_3 \in \{1, 2, \ldots, 9\}$. A quick computation with Maple reveals no solutions in the interval $n \in [10, 500]$. For this computation, we first note that Q_{500} has 192 digits. Thus, we generate the list of all numbers which are sums of at most 2 repdigits with at most 192 digits each, let us call it $\mathcal{A}$. Then, for every $n \in [10, 500]$, we compute $M := \lfloor \log Q_n / \log 10 \rfloor + 1$ (the number of digits of Q_n) and then check whether $Q_n - d\left(\frac{10^m-1}{9}\right)$ is a member of $\mathcal{A}$ for some digit $d \in \{1, \ldots, 9\}$ and some $m \in \{M-1, M\}$. This computation takes a few seconds. So, from now on, we assume that $n \ge 500$. We next comment on the size of m_1, m_2, m_3 versus n.

Lemma 4.1. *All solutions of equation* (27) *satisfy*

$$m_3 \log 10 - 4 < n\log\alpha < m_3 \log 10 + 2.$$

Proof. The proof follows easily from the fact that $\alpha^{n-1} < Q_n < \alpha^{n+1}$. One can see that

$$\alpha^{n-1} < Q_n < 3 \cdot 10^{m_3}.$$

Taking the logarithm on both sides, we get $(n-1)\log\alpha < \log 3 + m_3 \log 10$, which leads to

$$n\log\alpha < \log\alpha + \log 3 + m_3\log 10 < m_3 \log 10 + 2.$$

Similarly, the lower bound follows. □

4.2. Bounds of n, m_1, m_2, m_3

We next return to equation (27) and use the Binet formula (4) to get

$$\alpha^n + \beta^n = d_1\Big(\frac{10^{m_1}-1}{9}\Big) + d_2\Big(\frac{10^{m_2}-1}{9}\Big) + d_3\Big(\frac{10^{m_3}-1}{9}\Big).$$

Equation (27) can be expressed as

$$9(\alpha^n + \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} - d_3 10^{m_3} = -(d_1 + d_2 + d_3). \tag{28}$$

Here also, we examine (28) in three different steps as follows.
Step 1. Equation (28) gives

$$9\alpha^n - d_3 10^{m_3} = d_1 10^{m_1} + d_2 10^{m_2} - 9\beta^n - (d_1 + d_2 + d_3), \tag{29}$$

which we rewrite as

$$|9\alpha^n - d_3 10^{m_3}| = |d_1 10^{m_1} + d_2 10^{m_2} - 9\beta^n - (d_1 + d_2 + d_3)| < 54 \cdot 10^{m_2}.$$

Thus, dividing both sides by $d_3 10^{m_3}$, we get

$$\left| \left(\frac{9}{d_3} \right) \alpha^n 10^{-m_3} - 1 \right| < \frac{54}{10^{m_3 - m_2}}. \tag{30}$$

Let

$$\Gamma_1 := \left(\frac{9}{d_3} \right) \alpha^n 10^{-m_3} - 1. \tag{31}$$

Suppose that $\Gamma_1 = 0$. Then, we have

$$\alpha^n = \frac{d_3 10^{m_3}}{9},$$

which implies that $\alpha^{2n} \in \mathbb{Q}$, a contradiction. Thus, $\Gamma_1 \neq 0$. With the notations of Theorem 2.1, we take

$$\eta_1 = \frac{9}{d_3}, \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad b_1 = 1, \quad b_2 = n, \quad b_3 = -m_3.$$

Since $10^{m_3 - 1} < Q_n < \alpha^{n+1}$, we have $m_3 \leq n$. Therefore, we can take $D = n$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so $d_{\mathbb{L}} = 2$. We now need to take A_j for $j = 1, 2, 3$, such that

$$A_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}.$$

Note that

$$h(\eta_1) \leq h(9) + h(d_3) \leq h(9) + h(9) \leq 2h(9).$$

This implies that

$$2h(\eta_1) < 8.8.$$

Thus, we can take

$$A_1 := 8.8.$$

Clearly,

$$h(\eta_2) = \frac{1}{2} \log \alpha, \qquad h(\eta_3) = \log 10.$$

We have

$$\max\{2h(\eta_2), |\log \eta_2|, 0.16\} = \log \alpha < 0.9 := A_2, \tag{32}$$

$$\max\{2h(\eta_3), |\log \eta_3|, 0.16\} = 2 \log 10 < 4.7 := A_3. \tag{33}$$

Theorem 2.1 tells us that

$$\log |\Gamma_1| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (30) leads to

$$(m_3 - m_2) \log 10 < \log 54 + 3.7 \cdot 10^{13} (1 + \log n),$$

giving

$$m_3 - m_2 < 1.7 \cdot 10^{13}(1 + \log n). \tag{34}$$

<u>*Step 2*</u>. Equation (28) becomes

$$9\alpha^n - d_3 10^{m_3} - d_2 10^{m_2} = d_1 10^{m_1} - 9\beta^n - (d_1 + d_2 + d_3), \tag{35}$$

which we rewrite as

$$\left|9\alpha^n - 10^{m_2}(d_3 10^{m_3 - m_2} + d_2)\right| = |d_1 10^{m_1} - 9\beta^n - (d_1 + d_2 + d_3)| < 45 \cdot 10^{m_1}.$$

Thus, dividing both sides by $10^{m_2}(d_3 10^{m_3-m_2} + d_2)$, we get

$$\left|\left(\frac{9}{d_3 10^{m_3-m_2} + d_2}\right)\alpha^n 10^{-m_2} - 1\right| < \frac{45}{10^{m_2-m_1}}. \tag{36}$$

Let

$$\Gamma_2 := \left(\frac{9}{d_3 10^{m_3-m_2} + d_2}\right)\alpha^n 10^{-m_2} - 1. \tag{37}$$

Suppose that $\Gamma_2 = 0$. Then, we have

$$\alpha^n = \frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}.$$

Conjugating in $\mathbb{Q}(\alpha)$, we get

$$\beta^n = \frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}.$$

Consequently, we obtain

$$\frac{10^{m_3}}{9} \le \frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9} = |\beta|^n < 1,$$

which is a contradiction. Thus, $\Gamma_2 \neq 0$. To apply Theorem 2.1, we take

$$\eta_1 = \frac{9}{d_3 10^{m_3-m_2} + d_2}, \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad b_1 = 1, \quad b_2 = n, \quad b_3 = -m_2.$$

Again we take $D = n$. Furthermore, we have

$$\begin{aligned} h(\eta_1) &= h\left(\frac{9}{d_3 10^{m_3-m_2} + d_2}\right) \\ &\le h(9) + h(d_3 10^{m_3-m_2} + d_2) \\ &\le h(9) + h(d_3) + h(d_2) + (m_3 - m_2)h(10) + \log 2 \\ &\le 7.3 + 2.4(m_3 - m_2). \end{aligned}$$

That is,

$$2h(\eta_1) < 14.6 + 4.8(m_3 - m_2).$$

Thus, we take

$$A_1 = 14.6 + 4.8(m_3 - m_2).$$

Since η_2, η_3 are the same as in Γ_1, we use the same values for A_2, A_3. From Theorem 2.1, we obtain

$$\log|\Gamma_2| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (36) leads to

$$(m_2 - m_1)\log 10 < \log 45 + 4.11 \cdot 10^{12}(14.6 + 4.8(m_3 - m_2))(1 + \log n).$$

Hence, using inequality (34), we obtain

$$(m_2 - m_1)\log 10 < \log 45 + 4.11 \cdot 10^{12}(14.6 + 4.8(1.7 \cdot 10^{13}(1 + \log n)))(1 + \log n).$$

The above inequality gives us

$$m_2 - m_1 < 1.5 \cdot 10^{26}(1 + \log n)^2. \tag{38}$$

Step 3. Equation (28) becomes

$$9\alpha^n - d_3 10^{m_3} - d_2 10^{m_2} - d_1 10^{m_1} = -9\beta^n - (d_1 + d_2 + d_3), \tag{39}$$

which we rewrite as

$$\left|\alpha^n - 10^{m_3}(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)/9\right| = |-\beta^n - (d_1 + d_2 + d_3)/9| < 4.$$

Thus, dividing both sides by α^n, we get

$$\left|1 - \alpha^{-n}10^{m_3}(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)/9\right| < \frac{1}{\alpha^{n-1.6}}. \tag{40}$$

Put

$$\Gamma_3 := 1 - \alpha^{-n}10^{m_3}(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)/9. \tag{41}$$

The fact that $\Gamma_3 \neq 0$ can be justified by a similar argument as the fact that $\Gamma_2 \neq 0$. In order to apply Theorem 2.1, we take

$$\eta_1 = \frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9},$$

$$\eta_2 = \alpha, \quad \eta_3 = 10, \quad b_1 = 1, \quad b_2 = -n, \quad b_3 = m_3.$$

We have $D = n$, and A_2 and A_3 are as in (32) and (33). As for A_1, we have

$$\begin{aligned} h(\eta_1) &= h\Big(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\Big) \\ &\leq h\Big(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_2} + d_3}{9}\Big) \\ &\leq h(9) + h(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_2} + d_3) \\ &\leq h(9) + h(d_1) + h(d_2) + h(d_3) + (m_3 - m_2)h(10) \\ &\quad + (m_2 - m_1)h(10) + 2\log 2 \\ &\leq 10.2 + 2.4(m_3 - m_2) + 2.4(m_2 - m_1). \end{aligned}$$

That is,

$$2h(\eta_1) < 20.4 + 4.8(m_3 - m_2) + 4.8(m_2 - m_1).$$

Thus, in order to use inequalities (34) and (38), we take

$$A_1 = 20.4 + 4.8(m_3 - m_2) + 4.8(m_2 - m_1).$$

With Theorem 2.1, we get

$$\log|\Gamma_4| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (40) leads to

$$n\log\alpha - \log(\alpha^{1.6}) < 4.11\cdot 10^{12}(20.4 + 4.8(m_3 - m_2) + 4.8(m_2 - m_1))(1 + \log n).$$

Hence, using inequalities (34) and (38), we obtain

$$\begin{aligned} n\log\alpha - \log(\alpha^{1.6}) < 4.11\cdot 10^{12}(20.4 + 4.8(1.7\cdot 10^{13}(1 + \log n)) \\ + 4.8(1.5\cdot 10^{26}(1+\log n)^2))(1 + \log n). \end{aligned}$$

The above inequality gives us

$$n < 4\cdot 10^{45}.$$

Lemma 4.1 implies

$$m_1 \le m_2 \le m_3 < 1.74\cdot 10^{45}.$$

We summarize what we have proved so far in the following lemma.

Lemma 4.2. *All solutions of equation* (27) *satisfy*

$$m_1 \le m_2 \le m_3 < 1.74\cdot 10^{45}, \qquad n < 4\cdot 10^{45}.$$

4.3. Bound reduction

To lower the above bounds, we return to equation (27). We rewrite it into the form

$$Q_n = \frac{d_3 10^{m_3}}{9} + \left(d_1\frac{10^{m_1}-1}{9} + d_2\frac{10^{m_2}-1}{9} - \frac{d_3}{9}\right).$$

Observe that the term in parentheses is always positive since

$$\left(d_1\frac{10^{m_1}-1}{9} + d_2\frac{10^{m_2}-1}{9} - \frac{d_3}{9}\right) \ge 2\frac{10^{m_1}-1}{9} - \frac{1}{9} \ge 2 - \frac{1}{9} \ge \frac{7}{4} > 0.$$

Hence, we have

$$\alpha^n - \frac{d_3 10^{m_3}}{9} = \left(d_1\frac{10^{m_1}-1}{9} + d_2\frac{10^{m_2}-1}{9} - \frac{d_3}{9}\right) - \beta^n \ge \frac{7}{4} - \frac{1}{\alpha^{500}} > 0.$$

Thus, the number Γ_1 from (31) appearing inside the absolute value in inequality (30) is positive. Hence, with the above notations, we have

$$\alpha^n - \frac{d_3 10^{m_3}}{9} = \frac{d_3 10^{m_3}}{9}\left(e^{\Lambda_1} - 1\right) > 0,$$

where

$$\Lambda_1 = n\log\eta_2 - m_3\log\eta_3 + \log\eta_1.$$

Therefore, we obtain

$$0 < \Lambda_1 < \exp(\Lambda_1) - 1 = \Gamma_1 < \frac{54}{10^{m_3-m_2}},$$

which implies that

$$\begin{aligned} 0 &< \log\left(\frac{9}{d_3}\right) + m_3(-\log 10) + n\log\alpha \\ &< \frac{54}{10^{m_3-m_2}} < 10^{1.74}\exp(-2.3\cdot(m_3 - m_2)). \end{aligned}$$

Thus,

$$\Lambda_1 < 10^{1.74}\exp(-2.30\cdot(m_3 - m_2)),$$

with $Y := m_3 - m_2 < 1.74 \cdot 10^{45}$.

Therefore, to apply Lemma 2.3, we take

$$c = 10^{1.74}, \qquad \delta = 2.3, \qquad X_0 = 1.74 \cdot 10^{45}, \qquad \psi = \frac{\log(9/d_3)}{\log 10},$$
$$\vartheta = -\frac{\log \alpha}{\log 10}, \qquad \vartheta_1 = \log \alpha, \qquad \vartheta_2 = \log 10, \qquad \beta = \log(9/d_3).$$

The smallest value of $q > X_0$ is $q = q_{100}$. We find that q_{101} satisfies the hypothesis of Lemma 2.3. Applying Lemma 2.3, we get $m_3 - m_2 \le 53$ for all $d_3 = 1, 2, \ldots, 8$.

When $d_3 = 9$, we get that $\beta = 0$. The largest partial quotient a_k for $0 \le k \le 201$ is $a_{181} = 1556$. Applying Lemma 2.2, $m_3 - m_2 = Y < m_3 \le X_0 := 1.74 \cdot 10^{45}$ that

$$m_3 - m_2 < \frac{1}{2.3} \log\Big(\frac{10^{1.74}(1556+2) \cdot 1.74 \cdot 10^{45}}{|\log 10|}\Big).$$

We obtain $m_3 - m_2 \le 50$, so we get the same conclusion as before, namely, $m_3 - m_2 \le 53$.

We now take $0 \le m_3 - m_2 \le 53$. Let

$$\Lambda_2 = n \log \eta_2 - m_2 \log \eta_3 + \log \eta_1.$$

From equation (28), we have that

$$\frac{d_3 10^{m_3} + d_2 10^{m_2}}{9}(\mathrm{e}^{\Lambda_2} - 1) = -\beta^n + d_1 \frac{10^{m_1} - 1}{9} - \Big(\frac{d_3 + d_2}{9}\Big)$$
$$> -\frac{(-1)^n}{\alpha^n} + \frac{10^{m_1}}{9} - \frac{1}{3}.$$

Furthermore, we see

$$-\frac{(-1)^n}{\alpha^n} + \frac{10^{m_1}}{9} - \frac{1}{3} > -\frac{1}{\alpha^n} + \frac{7}{9} > -\frac{1}{\alpha^{500}} + \frac{7}{9} > 0.$$

Thus,

$$\mathrm{e}^{\Lambda_2} - 1 > 0.$$

So, from (35), we see that

$$\alpha^n - \frac{d_3 10^{m_3}}{9} - \frac{d_2 10^{m_2}}{9} = \Big(\frac{d_3 10^{m_3}}{9} + \frac{d_2 10^{m_2}}{9}\Big)\big(\mathrm{e}^{\Lambda_2} - 1\big) > 0,$$

then

$$0 < \Lambda_2 < \mathrm{e}^{\Lambda_2} - 1 = \Gamma_2 < \frac{45}{10^{m_2 - m_1}},$$

which implies

$$0 < \log\Big(\frac{9}{d_3 10^{m_3 - m_2} + d_2}\Big) + m_2(-\log 10) + n \log \alpha$$
$$< \frac{45}{10^{m_2 - m_1}} < 10^{1.66} \exp(-2.3 \cdot (m_2 - m_1)).$$

Thus, we get

$$\Lambda_2 < 10^{1.66} \exp(-2.3 \cdot (m_2 - m_1)),$$

with $Y := m_2 - m_1 < 1.74 \cdot 10^{45}$.

Therefore, in order to apply Lemma 2.3, we take

$$c = 10^{1.66}, \qquad \delta = 2.3, \qquad X_0 = 1.74 \cdot 10^{45}, \quad \psi = \frac{\log\left(\frac{9}{d_3 10^{m_3-m_2}+d_2}\right)}{\log 10},$$
$$\vartheta = -\frac{\log\alpha}{\log 10}, \quad \vartheta_1 = \log\alpha, \quad \vartheta_2 = \log 10, \qquad \beta = \log\left(\frac{9}{d_3 10^{m_3-m_2}+d_2}\right).$$

We get $q = q_{106} > X_0$. By Lemma 2.3, over all the possibilities for the digits $d_2, d_3 \in \{1, \ldots, 9\}$ and $m_3 - m_2 \in \{1, \ldots, 53\}$ except for $m_3 = m_2$ and $d_2 + d_3 = 9$, we get $m_2 - m_1 \leq 56$.

In the exceptional cases $m_3 = m_2$ and $d_3 + d_2 = 9$, one actually gets that $\beta = 0$. The largest partial quotient a_k for $0 \leq k \leq 201$ is $a_{181} = 1556$. Applying Lemma 2.2 with $m_2 - m_1 = Y < m_2 \leq X_0 := 1.74 \cdot 10^{45}$,

$$m_2 - m_1 < \frac{1}{2.3}\log\left(\frac{10^{1.66}(1556+2)\cdot 1.74\cdot 10^{45}}{|\log 10|}\right),$$

we obtain $m_2 - m_1 \leq 50$. So we get the same conclusion as before, namely $m_2 - m_1 \leq 56$.

We now take $0 \leq m_3 - m_1 \leq 109$ and $0 \leq m_3 - m_2 \leq 53$. Let

$$\Lambda_3 = m_3 \log\eta_3 - n\log\eta_2 + \log\eta_1.$$

From equation (28), we have

$$\alpha^n(1 - e^{\Lambda_3}) = -\beta^n - (d_1+d_2+d_3)/9 = -(\beta^n + (d_1+d_2+d_2)/9).$$

Furthermore,

$$\beta^n + (d_1+d_2+d_3)/9 > -\frac{1}{\alpha^n} + \frac{1}{3} > -\frac{1}{\alpha^{500}} + \frac{1}{3} > 0.$$

Thus,

$$e^{\Lambda_3} - 1 > 0.$$

So

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = |\Gamma_3| < \frac{4}{\alpha^n} < \frac{1}{\alpha^{n-1.6}}$$

which implies that

$$0 < \log\left(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\right) + m_3\log 10 + n(-\log\alpha)$$
$$< \frac{4}{\alpha^n} < \alpha^{1.6}\exp(-0.88\cdot n).$$

We keep the value for $X_0 = 4\cdot 10^{45}$, and only change ψ to

$$\psi = \log\left(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\right)/\log 10,$$

$$c = \alpha^{1.6}, \qquad \delta = 0.88, \qquad v = \frac{\log\alpha}{\log 10}, \qquad v_2 = \log 10, \qquad v_1 = -\log\alpha,$$
$$\beta = \log\left(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\right).$$

We get $q = q_{101} > X_0$, and by Lemma 2.3, we get $n \leq 203$. This holds for all choices of $d_1, d_2, d_3 \in \{1, \ldots, 9\}$, $m_3 - m_2 \in [0, 53]$ and $m_3 - m_1 \in [0, 109]$ except when $m_1 = m_2 = m_3$ and $d_1 + d_2 + d_3 = 9$, or when $m_3 - m_1 = m_3 - m_2 = 1$ and $d_1 + d_2 = 10$, $d_3 = 8$ cases in which $\beta = 0$.

For the cases when $\beta = 0$, the largest partial quotient a_k for $0 \leq k \leq 203$ is $a_{180} = 1556$. Applying again Lemma 2.2 with $n = Y \leq X_0 := 4 \cdot 10^{45}$, we get

$$n < \frac{1}{0.88} \log\Big(\frac{\alpha^{1.6}(1556+2)\cdot 4\cdot 10^{45}}{|\log 10|}\Big),$$

which leads to $n \leq 129$. Thus, we get the same conclusion as before, namely $n \leq 203$. But this contradicts the assumption that $n \geq 500$. Hence, the theorem is proved.

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