Acta Math. Univ. Comenianae
Vol. XC, 1 (2021), pp. 93–97

ON MEIR-KEELER TYPE CONTRACTION VIA RATIONAL EXPRESSION

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Abstract. In this paper, we show that the continuity requirement assumed in the main result of Vara Prasad and Singh [*Meir-Keeler type contraction via rational expression*, Acta Math. Univ. Comenianae LXXXIX(1) (2020), 19–25] can be relaxed further. As a by-product we explore some new answers to the open question posed by Rhoades [*Contemporary Mathematics* **72** (1988), 233–245] regarding the existence of contractive mappings that admit discontinuity at the fixed point.

1. Introduction

In [**11**], the authors proved the following theorem.

Theorem 1.1. *Let S be a continuous self-mapping on a complete metric space (X, d), we assume that the following condition satisfies*

(1)
$$\varepsilon \leq \phi\bigg(\max\left\{ \frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Sy)}{d(x,y)}, d(x,y) \right\}\bigg) < \varepsilon + \lambda(\varepsilon)$$
$$\implies d(Sx,Sy) < \varepsilon$$

for all $x, y \in X$, $x \neq y$ or $y \neq Sy$, where $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotonic increasing mapping, $\phi(t) < t$ for all $t > 0$, and $\phi(0) = 0$. Then S has a unique fixed point $\xi \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to ξ.

We now recall definitions of some weaker forms of continuity.

Definition 1.2 ([**3**]). A self-mapping S of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots$, if $S^k x_n \to Su$ whenever $\{x_n\}$ is a sequence in X such that $S^{k-1}x_n \to u$.

Definition 1.3 ([**2**]). If S is a self-mapping of a metric space (X, d), then the set $O(x, S) = \{S^n x : n = 1, 2, \ldots\}$ is called the orbit of S at x and S is called orbitally continuous if $u = lim_i S^{m_i} x$ implies $Su = lim_i SS^{m_i} x$.

Received April 2, 2020; revised May 1, 2020.
2010 *Mathematics Subject Classification.* Primary 47H09, 54E50; Secondary 47H10, 54E40.
Key words and phrases. Fixed point; $(\varepsilon - \delta)$ rational contraction; discontinuous mappings; k-continuity.

Definition 1.4 ([**4**])**.** A self-mapping S of a metric space (X, d) is called weakly orbitally continuous if the set $\{y \in X : lim_i S^{m_i} y = u \implies lim_i S S^{m_i} y = Su\}$ is nonempty whenever the set $\{x \in X : lim_i S^{m_i} x = u\}$ is nonempty.

Remark. The following observations are evident from Examples 1.5–1.8 [**4, 5**]

(i) 1- continuity is equivalent to continuity and

$$\text{continuity} \implies \text{2-continuity} \implies \text{3-continuity} \implies \dots,$$

but not conversely.

(ii) Continuity implies orbital continuity, but not conversely.

(iii) Orbital continuity implies weak orbital continuity, but the converse need not be true.

(iv) k-continuous mappings are orbitally continuous but the converse need not be true.

Example 1.5. Let $X = [0, 2]$ equipped with the usual metric and $S\colon X \to X$ be defined by

$$Sx = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x \le 2. \end{cases}$$

Then $Sx_n \to u \implies S^2 x_n \to u$ since $Sx_n \to u$ implies $u = 0$ or $u = 1$ and $S^2 x_n = 1$ for all n, that is, $S^2 x_n \to 1 = Su$. Hence S is 2-continuous and orbitally continuous. However, S is discontinuous at $x = 1$.

Example 1.6. Let $X = [0, 4]$ equipped with the usual metric. Define $S\colon X \to X$ by

$$Sx = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x \le 3, \\ \frac{x}{3} & \text{if } 3 < x \le 4. \end{cases}$$

Then $S^2 x_n \to u \implies S^3 x_n \to Su$ since $S^2 x_n \to u$ implies $u = 0$ or $u = 1$ and $S^3 x_n = 1 = Su$ for each n. Hence S is 3-continuous. However, $Sx_n \to u$ does not imply $S^2 x_n \to Su$, that is, S is not 2-continuous.

Example 1.7. Let $X = [0, 2]$ equipped with the usual metric. Define $S\colon X \to X$ by

$$Sx = \begin{cases} \frac{(1+3x)}{4} & \text{if } 0 \le x < 1, \\ 0 & \text{if } 1 \le x < 2, \\ 2 & \text{if } x = 2. \end{cases}$$

Then $S^n 0 \to 1$ and $S(S^n 0) \to 1 \ne S1$. Therefore, S is not orbitally continuous. However, S is weakly orbitally continuous. If we consider $x = 2$, then $S^n 2 \to 2$ and $S(S^n 2) \to 2 = S2$ and hence, S is weakly orbitally continuous. If we take the sequence $\{S^n 0\}$, then for any integer $k \ge 1$, we have $S^{k-1}(S^n 0) \to 1$ and $S^k(S^n 0) \to 1 \ne S1$. This shows that S is not k-continuous.

Example 1.8. Let $X = [0, \infty)$ equipped with the usual metric. Define $S : X \to X$ by

$$Sx = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ \frac{x}{5} & \text{if } x > 1. \end{cases}$$

Then S is orbitally continuous. Let $k \geq 1$ be any integer. Consider the sequence $\{x_n\}$ given by $x_n = 5^{k-1} + \frac{1}{n}$. Then $S^{k-1}x_n = 1 + \frac{1}{n5^{k-1}}, S^k x_n = \frac{1}{5} + \frac{1}{n5^k}$. This implies that $S^{k-1}x_n \to 1$ and $S^k x_n \to \frac{1}{5} \neq S1$ as $n \to \infty$. Hence S is not $k-$continuous.

2. Main results

Theorem 2.1. *Let S be a self-mapping on a complete metric space (X, d) satisfying* (1) *for all $x, y \in X$, $x \neq y$, or $y \neq Sy$. Suppose S is k-continuous for some $k \geq 1$ or S is orbitally continuous. Then S has a unique fixed point, say $\xi \in X$, and for each $x \in X$, the sequence of iterates $S^n x$ converges to the fixed point.*

Proof. It is obvious that S satisfies the following condition:

$$d(Sx, Sy) < \phi\left(\max\left\{ \frac{(1 + d(x, Sx))d(y, Sy)}{1 + d(x, y)}, \frac{d(x, Sx)d(y, Sy)}{d(x, y)}, d(x, y) \right\}\right). \tag{2}$$

Let x_0 be any point in X such that $x_0 \neq Sx_0$. Define the sequence $\{x_n\}$ in X recursively by $x_{n+1} = Sx_n$, i.e., $x_{n+1} = S^n x_0$ for some $n \in \mathbb{N} \cup \{0\}$. Following the proof of [**11**, Theorem 2.1], we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $\xi \in X$ such that $x_n \to \xi$ as $n \to \infty$. Also for each $k \geq 1$, we have $S^k x_n \to \xi$.

Now suppose that S is $k-$continuous. Since $S^{k-1}x_n \to \xi$, k-continuity of S implies that $\lim_{n\to\infty} S^k x_n = S\xi$. This yields $\xi = S\xi$, that is, ξ is a fixed point of S.

Finally, suppose that S is orbitally continuous. Since $\lim_{n\to\infty} x_n = \xi$, orbital continuity implies that $\lim_{n\to\infty} Sx_n = S\xi$. This gives $S\xi = \xi$, that is, ξ is a fixed point of S. Uniqueness of the fixed point follows from (2). $\square$

The following theorem improves the result of Radjel et al. [**8**]

Theorem 2.2. *Let S be a self-mapping on a complete metric space (X, d). We assume that the following condition satisfies*

$$\begin{aligned} &3\varepsilon \leq \left\{ \frac{(1 + d(x, Sx))d(y, Sy)}{1 + d(x, y)} + \frac{d(x, Sx)d(y, Sy)}{d(x, y)} + d(x, y) \right\} < 3\varepsilon + \lambda(\varepsilon) \\ &\Longrightarrow d(Sx, Sy) < \varepsilon \end{aligned} \tag{3}$$

for all $x, y \in X$, $x \neq y$ or $y \neq Sy$. Suppose S is k-continuous for some $k \geq 1$, or S is orbitally continuous. Then S has a unique fixed point, say $\xi \in X$, and for each $x \in X$, the sequence of iterates $S^n x$ converges to the fixed point.

Proof. Let x_0 be any point in X such that $x_0 \neq Sx_0$. Define the sequence $\{x_n\}$ in X recursively by $x_{n+1} = Sx_n$, i.e., $x_{n+1} = S^n x_0$ for some $n \in \mathbb{N} \cup \{0\}$. Following the proof given in [**8**], we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $\xi \in X$ such that $x_n \to \xi$ as $n \to \infty$. Also for each $k \geq 1$, we have $S^k x_n \to \xi$. Rest of the proof follows from the proof of Theorem 2.1. □

In the next theorem, we show that if a self-mapping S of a complete metric space (X, d) satisfies condition (1), then there exists a point, say z, in X such that for each $x \in X$, the sequence of iterates, i.e., $S^n x \to z$. However, z is a fixed point if and only if S is weakly orbitally continuous.

Theorem 2.3. *Let S be a self-mapping on a complete metric space (X, d) satisfying* (1) *for all $x, y \in X$, $x \neq y$ or $y \neq Sy$. Then S possesses a fixed point if and only if S is weakly orbitally continuous.*

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_{n+1} = Sx_n$, i.e., $x_{n+1} = S^n x_0$ for some $n \in \mathbb{N} \cup \{0\}$. Following the proof of [**11**, Theorem 2.1], we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also, for each integer $k \geq 1$, we have $S^k x_n \to z$ and using (2), $S^n y \to z$ for any $y \in X$.

Suppose that S is weakly orbitally continuous. Since $S^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of S, we get $S^n y_0 \to z$ and $S^{n+1} y_0 \to Sz$ for some $y_0 \in X$. This implies that $z = Sz$ since $S^{n+1} y_0 \to z$. Therefore, z is a fixed point of S.

Conversely, suppose that the mapping S possesses a fixed point, say z. Then $\{S^n z = z\}$ is a constant sequence such that $\lim_n S^n z = z$ and $\lim_n S^{n+1} z = z = Sz$. Hence, S is weakly orbitally continuous. Uniqueness of the fixed point follows easily. □

Remark. Theorems 2.1–2.3 give new solutions to the Rhoades problem [**9**] on the existence of contractive mappings that admit discontinuity at the fixed point. Some distinct answers of this problem are given in [**1, 3, 4, 5, 6, 7, 10**].

Acknowledgment. The author would like to thank the managing editor and the anonymous referee for their critical comments which helped to improve the manuscript.

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